that is $||f||_1 = \infty$ and thus $f \notin L^1(\mathbf{R}^n)$. On the other hand for every $\lambda > 0$

$$m(\{x \in \mathbf{R}^n \,:\, |f(x)| > \lambda\}) = m(B(0, \lambda^{-1/n})) = \frac{\Omega_n}{\lambda}$$

where Ω_n is a measure of a unit ball. Hence $f \in$ weak $L^1(\mathbf{R}^n)$.

Theorem 2.12 (Hardy-Littlewood I)**.** *If $f \in L^1(\mathbf{R}^n)$, then Mf is in weak $L^1(\mathbf{R}^n)$ and*

$$m(\{x \in \mathbf{R}^n \,:\, Mf(x) > \lambda\}) \le \frac{5^n}{\lambda} ||f||_1$$

for every $0 < \lambda < \infty$.

In other words, the maximal functions maps L^1 to weak L^1.
The proof of this theorem uses the Vitali covering theorem.

Theorem 2.13 (Vitali covering)**.** *Let $\mathcal{F}$ be a family of cubes Q s.t.*

$$\operatorname{diam}(\bigcup_{Q \in \mathcal{F}} Q) < \infty.$$

Then there exist a countable number of disjoint cubes $Q_i \in \mathcal{F}$, $i = 1, 2, \ldots$ s.t.

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{i=1}^{\infty} 5Q_i$$

Here $5Q_i$ is a cube with the same center as Q_i whose side length is multiplied by 5.

Proof. The idea is to choose cubes inductively at round i by first throwing away the ones intersecting the cubes $Q_1, \ldots, Q_{i-1}$ chosen at the earlier rounds and then choosing the largest of the remaining cubes not yet chosen. Because the largest cube was chosen at every round, it follows that $\cup_{j=1}^{i-1} 5Q_j$ will cover the cubes thrown away. However, implementing this intuitive idea requires some care because there can be infinitely many cubes in the family $\mathcal{F}$. In particular, it may not be possible to choose largest one, but we choose almost the largest one.

To work out the details, suppose that $Q_1, \ldots, Q_{i-1} \in \mathcal{F}$ are chosen. Define

$$l_i = \sup\{l(Q) \,:\, Q \in \mathcal{F} \text{ and } Q \cap \bigcup_{j=1}^{i-1} Q_j = \emptyset\}. \tag{2.14}$$

Observe first that $l_i < \infty$, due to $\operatorname{diam}(\bigcup_{Q \in \mathcal{F}} Q) < \infty$. If there is no a cube $Q \in \mathcal{F}$ such that

$$Q \cap \bigcup_{j=1}^{i-1} Q_j = \emptyset,$$

then the process will end and we have found the cubes $Q_1, \dots, Q_{i-1}$. Otherwise we choose $Q_i \in \mathcal{F}$ such that

$$l(Q_i) > \frac{1}{2} l_i \quad \text{and} \quad Q_i \cap \bigcup_{j=1}^{i-1} Q_j = \emptyset.$$

This is also how we choose the first cube. Observe further that this is possible since $0 < l_i < \infty$. We have chosen the cubes so that they are disjoint and it suffices to show the covering property.

Choose an arbitrary $Q \in \mathcal{F}$. Then it follows that this Q intersects at least one of the chosen cubes $Q_1, Q_2, \dots$, because otherwise

$$Q \cap Q_i = \emptyset \quad \text{for every} \quad i = 1, 2, \dots$$

and thus the sup in (2.14) must be at least $l(Q)$ so that

$$l_i \geq l(Q) \quad \text{for every} \quad i = 1, 2, \dots.$$

It follows that

$$l(Q_i) > \frac{1}{2} l_i \geq \frac{1}{2} l(Q) > 0$$

for every $i = 1, 2, \dots$, so that

$$m(\bigcup_{i}^{\infty} Q_i) = \sum_{i=1}^{\infty} m(Q_i) = \infty,$$

where we also used the fact that the cubes are disjoint. This contradicts the fact that $m(\bigcup_i^\infty Q_i) < \infty$ since $\bigcup_i^\infty Q_i$ is a bounded set according to assumption $\operatorname{diam}(\bigcup_{Q \in \mathcal{F}} Q) < \infty$. Thus we have shown that Q intersects a cube in Q_i, $i = 1, 2, \dots$. Then there exists a smallest index i so that

$$Q \cap Q_i \neq \emptyset.$$

implying

$$Q \cap \bigcup_{j=1}^{i-1} Q_j = \emptyset.$$

Furthermore, according to the procedure

$$l(Q) \leq l_i < 2l(Q_i)$$

and thus $Q \subset 5Q_i$ and moreover

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{i=1}^{\infty} 5Q_i. \qquad \square$$

Proof of Theorem 2.12. Remember the notation

$$E_\lambda = \{x \in \mathbf{R}^n : Mf(x) > \lambda\}, \quad \lambda > 0$$

so that $x \in E_\lambda$ implies that there exits a cube $Q_x \ni x$ such that

$$\fint_{Q_x} |f(y)| \, \mathrm{d}y > \lambda \tag{2.15}$$

If Q_x would cover E_λ, then the result would follow by the estimate

$$m(E_\lambda) \le m(Q) \le \int_{\mathbf{R}^n} \frac{|f(y)|}{\lambda} \, \mathrm{d}y.$$

However, this is not usually the case so we have to cover E_λ with cubes. But then the overlap of cubes needs to be controlled, and here we utilize the Vitali covering theorem.

In application of the Vitali covering theorem, there is also a technical difficulty that E_λ may not be bounded. This problem is treated by looking at the

$$E_\lambda \cap B(0,k).$$

Let $\mathcal{F}$ be a collection of cubes with the property (2.15), and $x \in E_\lambda \cap B(0,k)$. Now for every $Q \in \mathcal{F}$ it holds that

$$l(Q)^n = m(Q) < \frac{1}{\lambda} \int_Q |f(y)| \, \mathrm{d}y \le \frac{||f||_1}{\lambda},$$

so that

$$l(Q) \le \Big(\frac{||f||_1}{\lambda} \Big)^{1/n} < \infty.$$

Thus $\operatorname{diam}(\bigcup_{Q \in \mathcal{F}} Q) < \infty$ and the Vitali covering theorem implies

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{i=1}^{\infty} 5Q_i.$$

Combining the facts, we have

$$\begin{aligned} m(E_\lambda \cap B(0,k)) &\le m(\bigcup_{Q \in \mathcal{F}}^{\infty} Q) \le \sum_{i=1}^{\infty} m(5Q_i) = 5^n \sum_{i=1}^{\infty} m(Q_i) \\ &\overset{(2.15)}{\le} \frac{5^n}{\lambda} \sum_{i=1}^{\infty} \int_{Q_i} |f(y)| \, \mathrm{d}y \\ &\overset{\text{cubes are disjoint}}{=} \frac{5^n}{\lambda} \int_{\cup_{i=1}^{\infty} Q_i} |f(y)| \, \mathrm{d}y \le \frac{5^n}{\lambda} ||f||_1 . \end{aligned}$$

Then we pass to the original E_λ

$$m(E_\lambda) = \lim_{k \to \infty} m(E_\lambda \cap B(0,k)) \le \frac{5^n}{\lambda} ||f||_1 . \qquad \square$$

Remark 2.16. Observe that $f \in L^1(\mathbf{R}^n)$ implies that $Mf(x) < \infty$ a.e. $x \in \mathbf{R}^n$ because

$$\begin{aligned} m(\{x \in \mathbf{R}^n : Mf(x) = \infty\} &\le m(\{x \in \mathbf{R}^n : Mf(x) > \lambda\}) \\ &\le \frac{5^n}{\lambda} \|f\|_1 \to 0 \end{aligned}$$

as $\lambda \to \infty$.

Definition 2.17. (i)

$$f \in L^1(\mathbf{R}^n) + L^p(\mathbf{R}^n), \qquad 1 \le p \le \infty$$

if

$$f = g + h, \quad g \in L^1(\mathbf{R}^n), \quad h \in L^p(\mathbf{R}^n)$$

(ii)

$$T : L^1(\mathbf{R}^n) + L^p(\mathbf{R}^n) \to \text{ measurable functions}$$

is subadditive, if

$$|T(f+g)(x)| \le |Tf(x)| + |Tg(x)| \text{ a.e. } x \in \mathbf{R}^n$$

(iii) T is of strong type (p, p), $1 \le p \le \infty$, if there exists a constant C independent of functions $f \in L^p(\mathbf{R}^n)$ s.t.

$$\|Tf\|_p \le C \|f\|_p .$$

for every $f \in L^p(\mathbf{R}^n)$

(iv) T is of weak type (p, p), $1 \le p < \infty$, if there exists a constant C independent of functions $f \in L^p(\mathbf{R}^n)$ s.t.

$$m(\{x \in \mathbf{R}^n : Tf(x) > \lambda\}) \le \frac{C}{\lambda^p} \|f\|_p^p$$

for every $f \in L^p(\mathbf{R}^n)$.

Remark 2.18. (i) Observe that the maximal operator is subadditive, of weak type (1,1) that is

$$m(\{x \in \mathbf{R}^n : Mf(x) > \lambda\}) \le \frac{5^n}{\lambda} \|f\|_1 ,$$

of strong type (∞, ∞)

$$\|Mf\|_\infty \le C \|f\|_\infty ,$$

and *nonlinear*.

(ii) Strong (p, p) implies weak (p, p):

$$\begin{aligned} m(\{x \in \mathbf{R}^n : Tf(x) > \lambda\}) &\overset{\text{Chebysev}}{\le} \frac{1}{\lambda^p} \int_{\mathbf{R}^n} |Tf|^p \, \mathrm{d}x \\ &\overset{\text{strong } (p,p)}{\le} \frac{C}{\lambda^p} \int_{\mathbf{R}^n} |f|^p \, \mathrm{d}x. \end{aligned}$$

Theorem 2.19 (Hardy-Littlewood II)**.** *If $f \in L^p(\mathbf{R}^n)$, $1 < p \leq \infty$, then $Mf \in L^p(\mathbf{R}^n)$ and there exists $C = C(n,p)$ (meaning C depends on n, p) such that*

$$||Mf||_p \leq C\,||f||_p\,.$$

This is not true, when $p = 1$, cf. Example 2.3. The proof is based on the interpolation (Marcinkiewich interpolation theorem, proven below) between weak $(1,1)$ and strong (∞,∞). In the proof of the Marcinkiewich interpolation theorem, we use the following auxiliary lemma. 9.9.2010

Lemma 2.20. *Let $1 \leq p \leq q \leq \infty$. Then*

$$L^p(\mathbf{R}^n) \subset L^1(\mathbf{R}^n) + L^q(\mathbf{R}^n).$$

Proof. Let $f \in L^p(\mathbf{R}^n)$, $\lambda > 0$. We split f into two part as $f = f_1 + f_2$ by setting

$$f_1(x) = f\chi_{\{x\in\mathbf{R}^n\,:\,|f(x)|\leq\lambda\}}(x) = \begin{cases} f(x), & |f(x)| \leq \lambda \\ 0, & |f(x)| > \lambda, \end{cases}$$

$$f_2(x) = f\chi_{\{x\in\mathbf{R}^n\,:\,|f(x)|>\lambda\}}(x) = \begin{cases} f(x), & |f(x)| > \lambda \\ 0, & |f(x)| \leq \lambda. \end{cases}$$

We will show that $f_1 \in L^q$ and $f_2 \in L^1$

$$\begin{aligned} \int_{\mathbf{R}^n} |f_1(x)|^q \, \mathrm{d}x &= \int_{\mathbf{R}^n} |f_1(x)|^{q-p} \, |f_1(x)|^p \, \mathrm{d}x \\ &\overset{|f_1|\leq\lambda}{\leq} \lambda^{q-p} \int_{\mathbf{R}^n} |f_1(x)|^p \, \mathrm{d}x \\ &\overset{|f_1|\leq|f|}{\leq} \lambda^{q-p}\,||f||_p^p < \infty, \end{aligned}$$

$$\begin{aligned} \int_{\mathbf{R}^n} |f_2(x)| \, \mathrm{d}x &= \int_{\mathbf{R}^n} |f_2|^{1-p} \, |f_2|^p \, \mathrm{d}x \\ &\overset{|f_2|>\lambda \text{ or } f_2=0}{\leq} \lambda^{1-p} \int_{\mathbf{R}^n} |f_2|^p \, \mathrm{d}x \\ &\overset{|f_2|\leq|f|}{\leq} \lambda^{1-p}\,||f||_p^p < \infty. \end{aligned}$$

□

Theorem 2.21 (Marcinkiewicz interpolation theorem)**.** *Let $1 < q \leq \infty$,*

$$T : L^1(\mathbf{R}^n) + L^q(\mathbf{R}^n) \to \textit{measurable functions}$$

is subadditive, and

(i) T is of weak type $(1,1)$
(ii) T is of weak type (q,q), if $q < \infty$, and
T is of strong type (q,q), if $q = \infty$.