HARMONIC ANALYSIS

Contents

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1. Introduction

This lecture note contains a sketch of the lectures. More illustrations and examples are presented during the lectures.

The tools of the harmonic analysis have a wide spectrum of applications in mathematical theory. The theory has strong real world applications at the background as well:

- Signal processing: Fourier transform, Fourier multipliers, Singular integrals.
- Solving PDEs: Poisson integral, Hilbert transform, Singular integrals.
- Regularity of PDEs: Hardy-Littlewood maximal function, approximation by convolution, Calderón-Zygmund decomposition, BMO.

Example 1.1. *We consider a problem*

$$\Delta u = f \quad \text{in} \quad \mathbf{R}^n$$

where $f \in L^p(\mathbf{R}^n)$. The solution u is of the form

$$u(x) = C \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-2}} \, \mathrm{d}y.$$

One of the questions in the regularity theory of PDEs is, does u have the second derivatives in L^p i.e.

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbf{R}^n)?$$

If we formally differentiate u, we get

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = C \int_{\mathbf{R}^n} f(y) \underbrace{\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|x-y|^{n-2}}}_{|\cdot| \leq C/|x-y|^n} \, \mathrm{d}y.$$

It follows that $\int_{\mathbf{R}^n} f(y) \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|x-y|^{n-2}} \, \mathrm{d}y$ defines a singular integral $Tf(x)$. A typical theorem in the theory of singular integrals says

$$||Tf||_p \leq C \, ||f||_p$$

and thus we can deduce that $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbf{R}^n)$.

Example 1.2. *Suppose that we have three different signals f_1, f_2, f_3 with different frequencies but only one channel, and that we receive*

$$f = f_1 + f_2 + f_3$$

from the channel. The Fourier transform $\mathcal{F}(f)$ gives us a spectrum of the signal f with three spikes in $|\mathcal{F}(f)|$. We would like to recover the *signal f_1. Thus we take a multiplier (filter)*

$$a_1(y) := \chi_{(a,b)}(y) = \begin{cases} 1, & y \in (a,b), \\ 0, & \textit{otherwise}, \end{cases}$$

where the interval (a,b) contains the frequency of f_1. Thus formally by taking the inverse Fourier transform, we get

$$f_1 = \mathcal{F}^{-1}(a_1 \mathcal{F}(f)) =: Tf(x).$$

This, again formally, defines an operator T which turns out to be of the form

$$c \int_{\mathbf{R}} \frac{\sin(Cy)}{y} f(x-y)\,\mathrm{d}y$$

with some constants c, C. This operator is of a convolution type. However, $\sin(Cy)/y$ is not integrable over the whole $\mathbf{R}$, so this requires some care!

2. Hardy-Littlewood maximal function

Definition 2.1. Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and m a Lebesgue measure. A Hardy-Littlewood maximal function $Mf : \mathbf{R}^n \mapsto [0,\infty]$ is

$$Mf(x) = \sup_{Q \ni x} \frac{1}{m(Q)} \int_Q |f(y)|\,\mathrm{d}y =: \sup_{Q \ni x} \fint_Q |f(y)|\,\mathrm{d}y,$$

where the supremum is taken over all the cubes Q with sides parallel to the coordinate axis and that contain the point x. Above we used the shorthand notation

$$\fint_Q f(x)\,\mathrm{d}x = \frac{1}{m(Q)} \int_Q f(x)\,\mathrm{d}x$$

for the integral average.

Notation 2.2. We denote an open cube by

$$Q = Q(x,l) = \{y \in \mathbf{R}^n \ : \ \max_{1 \le i \le n} |y_i - x_i| < l/2\},$$

$$l(Q) \text{ is a side length of the cube } Q,$$

$$m(Q) = l(Q)^n,$$

$$\mathrm{diam}(Q) = l(Q)\sqrt{n}.$$

Example 2.3. $f : \mathbf{R} \to \mathbf{R},\ f(x) = \chi_{(0,1)}(x)$

$$Mf(x) = \begin{cases} \frac{1}{x}, & x > 1, \\ 1, & 0 \le x \le 1, \\ \frac{1}{1-x}, & x < 0. \end{cases}$$

Observe that $f \in L^1(\mathbf{R})$ but $Mf \notin L^1(\mathbf{R})$.

Remark 2.4. (i) Mf is defined at every point $x \in \mathbf{R}^n$ and if $f = g$ almost everywhere (a.e.), then $Mf(x) = Mg(x)$ at every $x \in \mathbf{R}^n$.

(ii) It may well be that $Mf = \infty$ for every $x \in \mathbf{R}^n$. Let for example $n = 1$ and $f(x) = x^2$.

(iii) There are several definitions in the literature which are often equivalent. Let

$$\tilde{M}f(x) = \sup_{l>0} \fint_{Q(x,l)} |f(y)| \, \mathrm{d}y,$$

where the supremum is taken over all cubes $Q(x, l)$ centered at x. Then clearly

$$\tilde{M}f(x) \leq Mf(x)$$

for all $x \in \mathbf{R}^n$. On the other hand, if Q is a cube such that $x \in Q$, then $Q = Q(x_0, l_0) \subset Q(x, 2l_0)$ and

$$\begin{aligned} \fint_Q |f(x)| \, \mathrm{d}y &\leq \frac{m(Q(x, 2l_0))}{m(Q(x, l_0))} \frac{1}{m(Q(x, 2l_0))} \int_{Q(x,2l_0)} |f(y)| \, \mathrm{d}y \\ &\leq 2^n \tilde{M}f(x) \end{aligned}$$

because

$$\frac{m(Q(x, 2l_0))}{m(Q(x, l_0))} = \frac{(2l_0)^n}{l_0^n} = 2^n.$$

It follows that $Mf(x) \leq 2^n \tilde{M}f(x)$ and

$$\tilde{M}f(x) \leq Mf(x) \leq 2^n \tilde{M}f(x)$$

for every $x \in \mathbf{R}^n$. We obtain a similar result, if cubes are replaced for example with balls.

Next we state some immediate properties of the maximal function. The proofs are left for the reader.

Lemma 2.5. *Let $f, g \in L^1_{loc}(\mathbf{R}^n)$. Then*

(i)

$$Mf(x) \geq 0 \text{ for all } x \in \mathbf{R}^n \text{ (positivity).}$$

(ii)

$$M(f + g)(x) \leq Mf(x) + Mg(x) \text{ (sublinearity)}$$

(iii)

$$M(\alpha f)(x) = |\alpha| \, Mf(x), \ \alpha \in \mathbf{R} \text{ (homogeneity).}$$

(iv)

$$M(\tau_y f) = (\tau_y Mf)(x) = Mf(x + y) \text{ (translation invariance).}$$

Lemma 2.6. *If $f \in C(\mathbf{R}^n)$, then*

$$|f(x)| \leq Mf(x)$$

for all $x \in \mathbf{R}^n$.

Proof. Let $f \in C(\mathbf{R}^n)$, $x \in \mathbf{R}^n$. Then

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } |f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta.$$

From this and the triangle inequality, it follows that

$$\left| \fint_Q |f(x)| \, \mathrm{d}y - |f(x)| \right| \overset{\fint_Q 1 \, \mathrm{d}y = 1}{=} \left| \fint_Q \Big(|f(y)| - |f(x)| \Big) \, \mathrm{d}y \right|$$
$$\leq \fint_Q ||f(y)| - |f(x)|| \, \mathrm{d}y \leq \fint_Q |f(y) - f(x)| \, \mathrm{d}y < \varepsilon$$

whenever $\mathrm{diam}(Q) = \sqrt{n}\, l(Q) < \delta$. Thus

$$|f(x)| = \lim_{Q \ni x, l(Q) \to 0} \fint_Q |f(x)| \, \mathrm{d}y \leq \sup_{Q \ni x} \fint_Q |f(x)| \, \mathrm{d}y = Mf(x). \qquad \square$$

Remember that $f : \mathbf{R}^n \to [-\infty, \infty]$ is *lower semicontinuous* if

$$\{x \in \mathbf{R}^n \, : \, f(x) > \lambda\} = f^{-1}((\lambda, \infty])$$

is open for all $\lambda \in \mathbf{R}$. Thus for example, χ_U is lower semicontinuous whenever $U \subset \mathbf{R}^n$ is open. It also follows that if f is lower semicontinuous then it is measurable.

Lemma 2.7. *Mf is lower semicontinuous and thus measurable.*

Proof. We denote

$$E_\lambda = \{x \in \mathbf{R}^n \, : \, Mf(x) > \lambda\}, \ \lambda > 0.$$

Whenever $x \in E_\lambda$ it follows that there exists $Q \ni x$ such that

$$\fint_Q |f(y)| \, \mathrm{d}y > \lambda.$$

Further

$$Mf(z) \geq \fint_Q |f(y)| \, \mathrm{d}y > \lambda$$

for every $z \in Q$, and thus

$$Q \subset E_\lambda. \qquad \square$$

Lemma 2.8. *If $f \in L^\infty(\mathbf{R}^n)$, then $Mf \in L^\infty(\mathbf{R}^n)$ and*

$$||Mf||_\infty \leq ||f||_\infty .$$

Proof.

$$\fint_{Q(x)} |f(y)| \, \mathrm{d}y \leq ||f||_\infty \fint_Q 1 \, \mathrm{d}x = ||f||_\infty ,$$

for every $x \in \mathbf{R}^n$. From this it follows that

$$||Mf||_\infty \leq ||f||_\infty . \qquad \square$$

Lemma 2.9. *Let E be a measurable set. Then for each $0 < p < \infty$, we have*

$$\int_E |f(x)|^p \, \mathrm{d}x = p \int_0^\infty \lambda^{p-1} m(\{x \in E : |f(x)| > \lambda\}) \, \mathrm{d}\lambda$$

Proof. Sketch:

$$\begin{aligned} \int_E |f(x)|^p \, \mathrm{d}x &= \int_{\mathbf{R}^n} \chi_E(x) p \int_0^{|f(x)|} \lambda^{p-1} \, \mathrm{d}\lambda \, \mathrm{d}x \\ &\overset{\text{Fubini}}{=} p \int_0^\infty \lambda^{p-1} \int_{\mathbf{R}^n} \chi_{\{x \in E : |f(x)| > \lambda\}}(x) \, \mathrm{d}x \, \mathrm{d}\lambda \\ &= p \int_0^\infty \lambda^{p-1} m(\{x \in E : |f(x)| > \lambda\}) \, \mathrm{d}\lambda. \end{aligned}$$

□

Definition 2.10. Let $f : \mathbf{R}^n \to [-\infty, \infty]$ be measurable. The function f belongs to weak $L^1(\mathbf{R}^n)$ if there exists a constant C such that $0 \leq C < \infty$ such that

$$m(\{x \in \mathbf{R}^n : |f(x)| > \lambda\}) \leq \frac{C}{\lambda}$$

for all $\lambda > 0$.

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Remark 2.11. (i) $L^1(\mathbf{R}^n) \subset$ weak $L^1(\mathbf{R}^n)$ because

$$\begin{aligned} m(\{x \in \mathbf{R}^n : |f(x)| > \lambda\}) &= \int_{\{x \in \mathbf{R}^n : |f(x)| > \lambda\}} 1 \, \mathrm{d}x \\ &\leq \int_{\{x \in \mathbf{R}^n : |f(x)| > \lambda\}} \underbrace{\frac{|f(x)|}{\lambda}}_{\geq 1} \, \mathrm{d}x \leq \frac{||f||_1}{\lambda}, \end{aligned}$$

for every $\lambda > 0$.

(ii) weak $L^1(\mathbf{R}^n)$ is not included into $L^1(\mathbf{R}^n)$. This can be seen by considering

$$f : \mathbf{R}^n \to [0, \infty], \; f(x) = |x|^{-n}.$$

Indeed,

$$\begin{aligned} \int_{B(0,1)} |f(x)| \, \mathrm{d}x = \int_{B(0,1)} |x|^{-n} \, \mathrm{d}x &= \int_0^1 \int_{\partial B(0,r)} r^{-n} \, \mathrm{d}S(x) \, \mathrm{d}r \\ &= \int_0^1 r^{-n} \underbrace{\int_{\partial B(0,r)} 1 \, \mathrm{d}S(x)}_{\omega_{n-1} r^{n-1}} \, \mathrm{d}r \\ &= \omega_{n-1} \int_0^1 \frac{1}{r} \, \mathrm{d}r = \infty, \end{aligned}$$