Lemma 5.13. *If $f, g \in S(\mathbf{R})$, then*

$$\widehat{f * g} = \hat{f}\hat{g}$$

Proof. The proof is based on Fubini's theorem. To this end, observe that by the proof of Young's inequality for convolution, Theorem 3.2, we have

$$\int_{\mathbf{R}}\int_{\mathbf{R}} \left|f(y)g(x-y)\ e^{-2\pi i x\xi}\right| \,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbf{R}} |f(y)| \int_{\mathbf{R}} |g(x-y)| \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Now we can calculate

$$\begin{aligned}
\widehat{f * g} &= \int_{\mathbf{R}}\int_{\mathbf{R}} f(y)g(x-y)\,\mathrm{d}y\ e^{-2\pi i x\xi}\,\mathrm{d}x \\
&\overset{\text{Fubini}}{=} \int_{\mathbf{R}} f(y) \int_{\mathbf{R}} g(x-y)e^{-2\pi i x\xi}\,\mathrm{d}x\,\mathrm{d}y \\
&\overset{x-y=z,\,\mathrm{d}x=\mathrm{d}z}{=} \int_{\mathbf{R}} f(y) \int_{\mathbf{R}} g(z)e^{-2\pi i(z+y)\xi}\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_{\mathbf{R}} f(y)e^{-2\pi i y\xi}\,\mathrm{d}y \int_{\mathbf{R}} g(z)e^{-2\pi i z\xi}\,\mathrm{d}z = \hat{f}\hat{g}.
\end{aligned}$$

□

Next we prove Plancherel's theorem. The theorem plays a central role, when extending the definition of the Fourier transform to the L^2-functions. It will also be needed in connection to singular integrals.

Theorem 5.14 (Plancherel)**.** *If $f \in S(\mathbf{R})$, then*

$$||f||_2 = ||\hat{f}||_2. \tag{5.15}$$

Proof. Set $g = \overline{\hat{f}}$. Then $\hat{g} = \overline{f}$. To see this, we first calculate

$$\begin{aligned}
g = \overline{\hat{f}} &= \overline{\int_{\mathbf{R}} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x} \\
&= \int_{\mathbf{R}} \overline{f}(x)e^{2\pi i x\xi}\,\mathrm{d}x \\
&= \int_{\mathbf{R}} \overline{f}(x)e^{-2\pi i x(-\xi)}\,\mathrm{d}x = \widehat{\overline{f}}(-\xi)
\end{aligned}$$

and thus by Corollary 5.12

$$\hat{g}(x) = F(\widehat{\overline{f}}(-\xi))(x) = \overline{f}(x).$$

Utilizing this and Lemma 5.9, we have

$$\begin{aligned}
||f||_2 &= \int_{\mathbf{R}} f(x)\overline{f}(x)\,\mathrm{d}x = \int_{\mathbf{R}} f(x)\hat{g}(x)\,\mathrm{d}x \\
&\overset{\textit{Lemma 5.9}}{=} \int_{\mathbf{R}} \hat{f}(x)g(x)\,\mathrm{d}x = \int_{\mathbf{R}} \hat{f}(x)\overline{\hat{f}}(x)\,\mathrm{d}x = ||\hat{f}||_2.
\end{aligned}$$

□

5.2. **On L^1.** As stated above for $f \in L^1(\mathbf{R})$, the Fourier transform $\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-2\pi ix\xi}\,\mathrm{d}x$ is well defined but it might well be that $\hat{f} \notin L^1(\mathbf{R})$.

Question: Then how do we obtain f from $\hat{f}$ in this case as $\int_{\mathbf{R}} \hat{f}(\xi)e^{2\pi ix\xi}\,\mathrm{d}\xi$ might not be well defined?

The answer is that we can make sure that the inversion formula makes sense by multiplying by a bumb function which makes sure that the integrand gets small enough values far away, and then pass to a limit.

Theorem 5.16. *Let $\phi \in L^1(\mathbf{R})$, be bounded and continuous with $\hat{\phi} \in L^1(\mathbf{R})$, $||\hat{\phi}||_1 = 1$. Then*

$$\lim_{\varepsilon\to 0}\left|\left|\int_{\mathbf{R}} \hat{f}(\xi)e^{2\pi ix\xi}\phi(-\varepsilon\xi)\,\mathrm{d}\xi - f(x)\right|\right|_1 = 0.$$

A suitable ϕ in the theorem above is for example $\phi(x) = e^{-\pi x^2}$, see Example 5.7.

Proof. First, we show that

$$\int_{\mathbf{R}} \hat{f}(\xi)e^{2\pi ix\xi}\phi(-\varepsilon\xi)\,\mathrm{d}\xi = (f * \hat{\phi}_\varepsilon)(x).$$

To this end, recall that $\widehat{\phi(-\varepsilon x)} = \hat{\phi}_\varepsilon(-\xi)$ and $\widehat{f(x)e^{2\pi ihx}} = \hat{f}(\xi - h)$ by Lemma 5.6. Observe that these results hold also for L^1 functions. Since ϕ is bounded also the proof of Lemma 5.9 holds. Thus

$$\begin{aligned}
\int_{\mathbf{R}} \hat{f}(\xi)e^{2\pi ix\xi}\phi(-\varepsilon\xi)\,\mathrm{d}\xi &= \int_{\mathbf{R}}\int_{\mathbf{R}} f(y)e^{-2\pi iy\xi}\,\mathrm{d}y\ e^{2\pi ix\xi}\phi(-\varepsilon\xi)\,\mathrm{d}\xi \\
&\overset{\text{Lemma 5.9}}{=} \int_{\mathbf{R}} f(y) \int_{\mathbf{R}} \big(e^{2\pi ix\xi}\phi(-\varepsilon\xi)\big)e^{-2\pi iy\xi}\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \int_{\mathbf{R}} f(y)F\big(e^{2\pi ix\xi}\phi(-\varepsilon\xi)\big)(y)\,\mathrm{d}y \qquad (5.17)\\
&\overset{\text{Lemma 5.6:(vi),(viii)}}{=} \int_{\mathbf{R}} f(y)\,\hat{\phi}_\varepsilon(x-y)\,\mathrm{d}y \\
&= (f * \hat{\phi}_\varepsilon)(x).
\end{aligned}$$

When dealing with convolutions, we showed in Theorem 3.7 that

$$(f * \hat{\phi}_\varepsilon)(x) \to f(x) \quad \text{in } L^1(R).$$

□

If $\hat{f} \in L^1(\mathbf{R})$, then the inversion formula $f(x) = \int_{\mathbf{R}} \hat{f}(\xi)e^{2\pi ix\xi}\,\mathrm{d}\xi$ works as such. This can be seen by adding a condition $\phi(0) = 1$ for the bumb function and passing to limit in (5.17) using Lebesgue's dominated convergence on the left.

5.3. **On L^2.**

Theorem 5.18. *Let $f \in L^2(\mathbf{R}^n)$, and $\phi_j \in S(\mathbf{R})$, $j = 1, 2, \ldots$ such that*

$$\lim_{j\to\infty} ||\phi_j - f||_2 = 0.$$

Then there exists a limit which we denote by $\hat{f}$ such that

$$\lim_{j\to\infty} ||\hat{\phi}_j - \hat{f}||_2 = 0.$$

The function $\hat{f}$ is called a Fourier transform of $f \in L^2(\mathbf{R})$.

Proof. First of all, there exists a sequence $\phi_j \in S(\mathbf{R})$, $j = 1, 2, \ldots$ such that

$$\lim_{j\to\infty} ||\phi_j - f||_2 = 0$$

because $S(\mathbf{R})$ is dense in $L^2(\mathbf{R})$: We have already seen that $C_0(\mathbf{R})$ is dense in $L^2(\mathbf{R})$. On the other hand, if $f \in C_0(\mathbf{R})$ then $C_0^\infty(\mathbf{R}) \ni f * \phi_\varepsilon \to f$ in $L^2(\mathbf{R})$, where ϕ_ε is a standard mollifier, and we see that $C_0^\infty(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, which is contained in $S(\mathbf{R})$.

Then by Plancherel's theorem

$$||\hat{\phi}_j - \hat{\phi}_k||_2 = ||\phi_j - \phi_k||_2 \to 0$$

as $j, k \to \infty$ and thus $\hat{\phi}_j$, $j = 1, 2, \ldots$ is a Cauchy sequence. Since $L^2(\mathbf{R})$ is complete, $\hat{\phi}_j$ converges to a limit, which we denote by $\hat{f}$.

Next we show that the limit is independent of the approximating sequence. Let φ_j be another sequence such that

$$\varphi_j \to f \quad \text{in} \quad L^2(\mathbf{R})$$

and let $g \in L^2(\mathbf{R})$ be the limit

$$\hat{\varphi}_j \to g \quad \text{in} \quad L^2(\mathbf{R}).$$

Then

$$0 \overset{\phi_j, \varphi_j \to f}{=} \lim_{j\to 0} ||\varphi_j - \phi_j||_2 \overset{\text{Plancherel}}{=} \lim_{j\to 0} ||\hat{\varphi}_j - \hat{\phi}_j||_2 = ||g - \hat{f}||_2. \qquad \square$$

Similarly we obtain a unique inverse Fourier transform of any L^2-function.

We state separately a result from the previous proof.

Corollary 5.19 (Plancerel in L^2)**.** *If $f \in L^2(\mathbf{R})$, then*

$$||f||_2 = ||\hat{f}||_2.$$

Proof.

$$||f||_2 = \lim_{j\to\infty} ||\phi_j||_2 = \lim_{j\to\infty} ||\hat{\phi}_j||_2 = ||\hat{f}||_2.$$

$\square$

We also obtain formulas for calculating the Fourier transform and the inverse Fourier transform for L^2-functions. Observe that in the corollary below, $\chi_{B(0,R)} f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ by Hölder's inequality since $⨍_B |f| \,\mathrm{d}x \le \left(⨍_B |f|^2 \,\mathrm{d}x \right)^{1/2}$.

Corollary 5.20. *If $f \in L^2(\mathbf{R})$, then*

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{\{|x|<R\}} f(x) e^{-2\pi i x \xi} \,\mathrm{d}x,$$

and

$$f(x) = \lim_{R\to\infty} \int_{\{|\xi|<R\}} \hat{f}(\xi) e^{2\pi i x \xi} \,\mathrm{d}\xi.$$

Proof. Recall that if $f \in L^2(\mathbf{R})$, then $\chi_{B(0,R)} f \to f$ in $L^2(\mathbf{R})$ by Lebesgue's monotone/dominated convergence theorem. Let us denote

$$\lim_{R\to\infty} \int_{\{|x|<R\}} f(x) e^{-2\pi i x \xi} \,\mathrm{d}x = \lim_{R\to\infty} F(f\chi_{B(0,R)}).$$

The convergence $F(f\chi_{B(0,R)}) \to \hat{f}$ follows from the Plancherel's theorem, because the right hand side of

$$\left|\left| F(f\chi_{B(0,R)}) - \hat{f} \right|\right|_2 = \left|\left| f\chi_{B(0,R)} - f \right|\right|_2$$

can be made as small as we please by choosing R large enough. The proof of the inversion formula is similar. □

5.4. **On L^p, $1 < p < 2$.** Fourier transform is a linear operator and thus for $f \in L^p(\mathbf{R})$, $1 < p < 2$, we have

$$f = f_1 + f_2 = f\chi_{\{|f|>\lambda\}} + f\chi_{\{|f|\le\lambda\}} \in L^1 + L^2.$$

we have $\hat{f} = \hat{f}_1 + \hat{f}_2 \in L^\infty + L^2$ and

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{\{|x|<R\}} f(x) e^{-2\pi i x \xi} \,\mathrm{d}x,$$

can also be utilized here. However by a special case of the *Riesz-Thorin interpolation theorem* we obtain even better. We omit the proof.

Theorem 5.21 (Riesz-Thorin interpolation)**.** *Let T be a linear operator*

$$T : L^1(\mathbf{R}) + L^2(\mathbf{R}) \to L^\infty(\mathbf{R}) + L^2(\mathbf{R})$$

such that

$$||Tf_1||_\infty \le C_1 ||f_1||_1$$

for every $f_1 \in L^1(\mathbf{R})$, and

$$||Tf_2||_2 \le C_2 ||f_2||_2 ,$$

for every $f_2 \in L^2(\mathbf{R})$. *Then*

$$||Tf||_{p'} \leq C_1^{1-2/p'} C_2^{2/p'} ||f||_p ,$$

where $1/p + 1/p' = 1$.

Corollary 5.22 (Hausdorff-Young inequality)**.** *If* $f \in L^p(\mathbf{R})$, $1 \leq p \leq 2$, *then* $\hat{f} \in L^{p'}(\mathbf{R})$ *and*

$$||\hat{f}||_{p'} \leq ||f||_p .$$

Proof. By Lemma 5.6, we have $\left|\left|\hat{f}\right|\right|_\infty \leq ||f||_1$ and by Plancherel's theorem $\left|\left|\hat{f}\right|\right|_2 = ||f||_2$. Thus we can use Riesz-Thorin interpolation. □

Observe however that obtaining f from $\hat{f}$ by using

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{\{|x|<R\}} f(x) e^{-2\pi i x\xi} \, \mathrm{d}x,$$

is a nontrivial problem. For example in the case $p = 1$ the Fourier transform of $\chi_{B(0,R)}$ is not in L^1 as shown in Example 5.3, it does not satisfy the assumptions of Theorem 5.16, and thus our results do not imply the convergence. In higher dimensions there is no, in general,
14.10.2010 the convergence in L^p, $p \neq 2$, as $R \to \infty$.