Example 5.3 (Warning)**.** *The Fourier transform is well defined for $f \in L^1(\mathbf{R})$ because*

$$\left|f(x)e^{-2\pi i x\xi}\right| = |f(x)|$$

which is integrable. However, nothing guarantees that $\hat{f}(\xi)$ would be in $L^1(\mathbf{R})$. Indeed let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = \chi_{\{-1/2,1/2\}}(x)$, which is in $L^1(\mathbf{R})$. Then for $\xi \neq 0$,

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbf{R}} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x \\
&= \int_{-1/2}^{1/2} e^{-2\pi i x\xi}\,\mathrm{d}x \\
&= \int_{-1/2}^{1/2} \cos(2\pi x\xi)\,\mathrm{d}x - i\underbrace{\int_{-1/2}^{1/2} \sin(2\pi x\xi)\,\mathrm{d}x}_{=0} \\
&= \Big/_{-1/2}^{1/2} \frac{\sin(2\pi x\xi)}{2\pi\xi} \\
&= \frac{2\sin(\pi\xi)}{2\pi\xi} = \frac{\sin(\pi\xi)}{\pi\xi},
\end{aligned}$$

but $\frac{\sin(\pi\xi)}{\pi\xi}$ is not integrable (the integral of the positive part $= \infty$ and the integral over the negative part $= -\infty$ over any interval $(a, \infty]$). Later, we would like to write

$$F^{-1}\hat{f}(\xi) = \int_{\mathbf{R}} \hat{f}(x)e^{2\pi i x\xi}\,\mathrm{d}x$$

for the inverse Fourier transform, which however makes no sense as such for the function that is not integrable.

The problem described in the example above does not appear for the functions that are smooth and decay rapidly at the infinity, the so called Schwartz class. Later we use the functions on the Schwartz class to define Fourier transform in L^2 and further in L^p.

Definition 5.4. A function f is in the Schwartz class $S(R)$ if

(i) $f \in C^\infty(\mathbf{R})$

(ii)

$$\sup_{x\in\mathbf{R}} |x|^k \left|\frac{d^l f(x)}{dx^l}\right| < \infty, \quad \text{for every} \quad k, l \geq 0.$$

In other words, every derivative decays at least as fast as any power of $|x|$.

Example 5.5. *The standard mollifier (as well as all of $C_0^\infty(\mathbf{R})$)*

$$\varphi = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right), & x \in (-1,1) \\ 0, & \textit{else.} \end{cases}$$

is in $S(\mathbf{R})$. Also for the Gaussian

$$f(x) = e^{-x^2} \in S(\mathbf{R}).$$

Indeed,

$$\frac{df(x)}{dx} = -2xe^{-x^2} = -2xf(x)$$

and so forth so that all the derivatives will be of the form

$$\textit{polynomial} \cdot f(x)$$

and

$$|x|^k \, |\textit{polynomial} \cdot f(x)| \leq |\textit{polynomial}| \, |f(x)| \, .$$

Thus as e^{-x^2} decays faster than any polynomial, we see that $e^{-x^2} \in S(\mathbf{R})$.

Lemma 5.6. *Suppose that $f \in S(\mathbf{R})$. Then*

(i) $\widehat{(\alpha f + \beta g)} = \alpha \hat{f} + \beta \hat{g}$.
(ii) $\widehat{\left(\frac{df}{dx}\right)}(\xi) = 2\pi i \xi \hat{f}(\xi)$.
(iii) $\frac{d\hat{f}}{d\xi}(\xi) = (\widehat{-2\pi i x f})(\xi)$,
(iv) $\hat{f}$ *is continuous,*
(v) $||\hat{f}||_\infty \leq ||f||_1$,
(vi) $\widehat{f(\varepsilon x)} = \frac{1}{\varepsilon}\hat{f}(\frac{\xi}{\varepsilon}) = \hat{f}_\varepsilon(\xi), \varepsilon > 0$,
(vii) $f\widehat{(x+h)} = \hat{f}(\xi)e^{2\pi i h \xi}$,
(viii) $f\widehat{(x)e^{2\pi i h x}} = \hat{f}(\xi - h)$,

Proof. (i) Integral is linear.
(ii)

$$\begin{aligned}
\widehat{\left(\frac{df}{dx}\right)}(\xi) &= \int_{\mathbf{R}} \left(\frac{df}{dx}\right) e^{-2\pi i x \xi} \, \mathrm{d}x \\
&\overset{\text{integrate by parts}}{=} -\int_{\mathbf{R}} f(x) \frac{d}{dx} e^{-2\pi i x \xi} \, \mathrm{d}x \\
&= 2\pi i \xi \int_{\mathbf{R}} f(x) e^{-2\pi i x \xi} \, \mathrm{d}x = 2\pi i \xi \hat{f}(\xi).
\end{aligned}$$

(iii)

$$\begin{aligned}\frac{d\hat{f}}{d\xi}(\xi) &= \frac{d}{d\xi}\int_{\mathbf{R}} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x\\ &= \int_{\mathbf{R}} f(x)\frac{d}{d\xi}e^{-2\pi i x\xi}\,\mathrm{d}x\\ &= -\int_{\mathbf{R}} f(x)2\pi i x e^{-2\pi i x\xi}\,\mathrm{d}x\\ &= (\widehat{-2\pi i x f})(\xi).\end{aligned}$$

The interchange of the derivative and integral is ok as $f \in S(\mathbf{R})$: in the detailed proof one can write down the difference quotient and estimate it by definition of $S(\mathbf{R})$.

(iv)

$$\begin{aligned}\lim_{h\to 0}\hat{f}(\xi+h) &= \lim_{h\to 0}\int_{\mathbf{R}} f(x)e^{-2\pi i x(\xi+h)}\,\mathrm{d}x\\ &\overset{\text{DOM, } \left|f(x)e^{-2\pi i x(xi+h)}\right|\le|f(x)|}{=} \int_{\mathbf{R}} f(x)\lim_{h\to 0}e^{-2\pi i x(\xi+h)}\,\mathrm{d}x = \hat{f}(\xi).\end{aligned}$$

(v)

$$\left|\int_{\mathbf{R}} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x\right| \le \int_{\mathbf{R}} |f(x)|\underbrace{\left|e^{-2\pi i x\xi}\right|}_{=1}\,\mathrm{d}x.$$

(vi)

$$\begin{aligned}\widehat{f(\varepsilon x)} &= \int_{\mathbf{R}} f(\varepsilon x)e^{-2\pi i x\xi}\,\mathrm{d}x\\ &\overset{y=\varepsilon x,\,\mathrm{d}y=\varepsilon\,\mathrm{d}x}{=} \frac{1}{\varepsilon}\int_{\mathbf{R}} f(y)e^{(-2\pi i y\xi)/\varepsilon}\,\mathrm{d}y = \frac{1}{\varepsilon}\hat{f}(\frac{\xi}{\varepsilon}).\end{aligned}$$

(vii)

$$\begin{aligned}f\widehat{(x+h)} &= \int_{\mathbf{R}} f(x+h)e^{-2\pi i x\xi}\,\mathrm{d}x\\ &\overset{y=x+h,\,\mathrm{d}y=\mathrm{d}x}{=} \int_{\mathbf{R}} f(y)e^{-2\pi i(y-h)\xi}\,\mathrm{d}y = \hat{f}(\xi)e^{2\pi i h\xi}.\end{aligned}$$

(viii)

$$\begin{aligned}f\widehat{(x)e^{2\pi i h x}} &= \int_{\mathbf{R}} f(x)e^{2\pi i h x}e^{-2\pi i x\xi}\,\mathrm{d}x\\ &= \int_{\mathbf{R}} f(x)e^{-2\pi i x(\xi-h)}\,\mathrm{d}x = \hat{f}(\xi-h).\end{aligned}$$

□

Example 5.7. *If*

$$f(x) = e^{-\pi x^2}$$

then its Fourier transform is

$$\hat{f}(\xi) = e^{-\pi\xi^2}$$

By using complex integration around a rectangle and recalling that $e^{-\pi z^2}$ *is analytic function, we could calculate* $\int_{\mathbf{R}} e^{-\pi x^2} e^{-2\pi i x\xi}\, \mathrm{d}x$ *directly by using complex integration. We however follow a strategy that does not require complex integration and observe that* $f(x) = e^{-\pi x^2}$ *solves the differential equation*

$$\begin{cases} f' + 2\pi x f = 0 \\ f(0) = 1. \end{cases}$$

By taking Fourier transform of $f' + 2\pi x f = 0$ *and using Lemma 5.6, we obtain*

$$0 = F(f' + 2\pi x f) = \widehat{f'} + \widehat{2\pi x f} = 2\pi i \xi \hat{f} - \frac{\hat{f}'}{i} = i(2\pi\xi\hat{f} + \hat{f}').$$

And

$$\hat{f}(0) = \int_{\mathbf{R}} e^{-\pi x^2}\, \mathrm{d}x = 1$$

because

$$\begin{aligned} \Big(\int_{\mathbf{R}} e^{-\pi x^2}\, \mathrm{d}x\Big)^2 &= \int_{\mathbf{R}}\int_{\mathbf{R}} e^{-\pi x^2} e^{-\pi x^2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_0^\infty \int_{\partial B(0,r)} e^{-\pi r^2}\, \mathrm{d}r\, \mathrm{d}S \\ &= \int_0^\infty 2\pi r e^{-\pi r^2}\, \mathrm{d}r \\ &= -\Big/_0^\infty e^{-\pi r^2} = 1. \end{aligned}$$

Thus $\hat{f}$ *satisfies the same differential equation and the uniqueness of such a solution implies the claim.*

Theorem 5.8. *If* $f \in S(\mathbf{R})$*, then*

(i) $\hat{f} \in S(\mathbf{R})$ *(similar result does not hold in* L^1*),*
(ii)

$$F^{-1}(f) := \int_{\mathbf{R}} f(\xi) e^{2\pi i x\xi}\, \mathrm{d}\xi \in S(\mathbf{R})$$

whenever $f \in S(\mathbf{R})$*.*

Proof. (i) Recall that by Lemma 5.6, $\hat{f}$ is continuous and for any pair of integers k, l

$$\begin{aligned} F\bigg(\frac{1}{(2\pi i)^k}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^k(-2\pi i x)^l f(x)\bigg) &= \frac{1}{(2\pi i)^k}F\bigg(\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^k(-2\pi i x)^l f(x)\bigg) \\ &= \frac{1}{(2\pi i)^k}(2\pi i\xi)^k F\Big((-2\pi i x)^l f(x)\Big) \\ &= \frac{1}{(2\pi i)^k}(2\pi i\xi)^k\Big(\frac{\mathrm{d}}{\mathrm{d}\xi}\Big)^l\hat{f}(\xi) \\ &= \xi^k\Big(\frac{\mathrm{d}}{\mathrm{d}\xi}\Big)^l\hat{f}(\xi). \end{aligned}$$

Therefore

$$\begin{aligned} |\xi|^k\left|\Big(\frac{\mathrm{d}}{\mathrm{d}\xi}\Big)^l\hat{f}(\xi)\right| &= \left|\xi^k\Big(\frac{\mathrm{d}}{\mathrm{d}\xi}\Big)^l\hat{f}(\xi)\right| \\ &= \left|F\bigg(\frac{1}{(2\pi i)^k}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^k(-2\pi i x)^l f(x)\bigg)\right| \\ &\overset{\text{Lemma 5.6}}{\leq} \left\|\frac{1}{(2\pi i)^k}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^k(-2\pi i x)^l f(x)\right\|_1 < \infty \end{aligned}$$

so that $\hat{f} \in S(\mathbf{R})$.

(ii) This follows from the previous by a change of variable. □

Lemma 5.9. *If $f, g \in S(\mathbf{R})$, then*

$$\int_{\mathbf{R}}\hat{f}(x)g(x)\,\mathrm{d}x = \int_{\mathbf{R}}f(x)\hat{g}(x)\,\mathrm{d}x$$

Proof.

$$\begin{aligned} \int_{\mathbf{R}}\hat{f}(y)g(y)\,\mathrm{d}y &= \int_{\mathbf{R}}\int_{\mathbf{R}}f(x)e^{-2\pi i x y}\,\mathrm{d}x\ g(y)\,\mathrm{d}y \\ &\overset{\text{Fubini}}{=} \int_{\mathbf{R}}f(x)\int_{\mathbf{R}}e^{-2\pi i x y}g(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbf{R}}f(x)\hat{g}(x)\,\mathrm{d}x. \end{aligned}$$ □

Next one of the main results of the section: inversion formula for the rapidly decreasing functions:

Theorem 5.10 (Fourier inversion)**.** *If $f \in S(\mathbf{R})$, then*

$$f(x) = \int_{\mathbf{R}}\hat{f}(y)e^{2\pi i x\xi}\,\mathrm{d}\xi,$$

or with the other notation $f(x) = F^{-1}(F(f)) = F^{-1}(\hat{f})$.

Proof. First we show that

$$f(0) = \int_{\mathbf{R}} \hat{f}(y)\,\mathrm{d}y. \tag{5.11}$$

To see this let $\phi \in S(\mathbf{R})$ and define $h(y) = f(-y)$. Then $\hat{\phi} \in S(\mathbf{R})$ and by the convergence result Theorem 3.12 (and the remark after the theorem)

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}} h(-y)\hat{\phi}_\varepsilon(y)\,\mathrm{d}y = \lim_{\varepsilon \to 0}(h * \hat{\phi}_\varepsilon)(0) = h(0) = f(0).$$

On the other hand, by Lemma 5.6 and the previous lemma

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_{\mathbf{R}} h(-y)\hat{\phi}_\varepsilon(y)\,\mathrm{d}y &= \lim_{\varepsilon \to 0} \int_{\mathbf{R}} \widehat{h(-y)}\phi(\varepsilon y)\,\mathrm{d}y \\ &\overset{h(-y)=f(y)}{=} \lim_{\varepsilon \to 0} \int_{\mathbf{R}} \hat{f}(y)\phi(\varepsilon y)\,\mathrm{d}y.\end{aligned}$$

Let $\phi(x) = e^{-\pi x^2}$, then

$$\lim_{\varepsilon \to 0} \phi(\varepsilon x) = 1, \qquad \left|\hat{f}(y)\phi(\varepsilon y)\right| \le \left|\hat{f}(\xi)\right|.$$

It follows that

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}} \hat{f}(y)\phi(\varepsilon y)\,\mathrm{d}y \overset{\mathrm{DOM}}{=} \int_{\mathbf{R}} \hat{f}(y) \underbrace{\lim_{\varepsilon \to 0} \phi(\varepsilon y)}_{=1}\,\mathrm{d}y$$

proving (5.11). Then defining $g(x) := f(x+h)$ and using from Lemma 5.6 the fact that $\hat{g}(y) = \widehat{f(x+h)} = \hat{f}(y)e^{2\pi hy}$ and observing $g(0) = f(h)$, the equation (5.11) implies

$$f(h) = \int_{\mathbf{R}} \hat{f}(y)e^{2\pi ihy}\,\mathrm{d}y,$$

which proves the claim. □ 12.10.2010

Corollary 5.12. *Let $f \in S(\mathbf{R})$. Then by taking consecutive Fourier transforms, we obtain*

$$f(x) \xrightarrow{F} \hat{f}(\xi) \xrightarrow{F} f(-x) \xrightarrow{F} \hat{f}(-\xi) \xrightarrow{F} f(x).$$

In particular, $F^{-1}(\hat{f}) = F(F(F(\hat{f})))$.

Proof. The second arrow:

$$\begin{aligned}\int_{\mathbf{R}} \hat{f}(\xi)e^{-2\pi ix\xi}\,\mathrm{d}\xi &\overset{\xi=-\zeta}{=} \int_{\mathbf{R}} \hat{f}(-\zeta)e^{2\pi ix\zeta}\,\mathrm{d}\zeta \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} f(y)e^{-2\pi iy(-\zeta)}\,\mathrm{d}y\; e^{2\pi ix\zeta}\,\mathrm{d}\zeta \\ &\overset{y=-z}{=} \int_{\mathbf{R}}\int_{\mathbf{R}} f(-z)e^{-2\pi iz\zeta}\,\mathrm{d}z\; e^{2\pi ix\zeta}\,\mathrm{d}\zeta = f(-x).\end{aligned}$$

The other arrows are easier. □