Coprime factorizations and stabilizability of infinite-dimensional linear systems

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Main Theorem

The following are equivalent for a holomorphic function P:

(i) P has a dynamic **stabilizing controller**.

(ii) P has a right **coprime factorization**. [Smith89] [M05d]

(iii) P has a **stabilizable and detectable realization**. [Staffans98] [CurOpm05] [M05c]

We work in discrete time, but essentially the same results hold in continuous time too. Part of the results are new even in the scalar-valued case.

As corollaries, one obtains analogous results for exponential (power) stabilization.

Notation

$\mathtt{U}, \mathtt{X}, \mathtt{Y}$: complex Hilbert spaces of arbitrary dimensions.

$\mathbb{D}$: the unit disc $\{z \in \mathbb{C} \,\big|\, |z| < 1\}$.

$\mathcal{B}(\mathtt{U}, \mathtt{Y})$: bounded linear maps $\mathtt{U} \to \mathtt{Y}$.

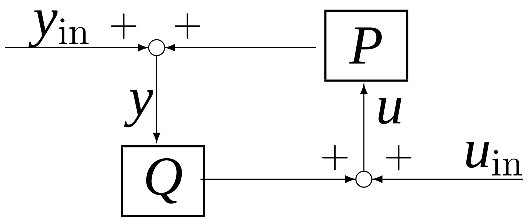
$\mathrm{H}^\infty(\mathtt{U}, \mathtt{Y})$: the set of bounded holomorphic functions $\mathbb{D} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$.

I: the identity operator, e.g., $I = I_{\mathtt{U}} \in \mathcal{B}(\mathtt{U}, \mathtt{U})$, or the corresponding constant function, e.g., $I = I_{\mathtt{U}} \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{U})$.

proper function $=$ holomorphic (operator-valued) function defined near the origin;
strictly proper $= P$ is proper and $P(0) = 0$;
stable $= \mathrm{H}^\infty$ (a restriction of a H^∞ function is identified with the H^∞ function).

Motivation: $P \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{Y}) \Longrightarrow P$ is bounded (stable) multiplication operator $\mathrm{H}^2(\mathtt{U}) \to \mathrm{H}^2(\mathtt{Y})$.

Dynamic (output-feedback) stabilization



$$\begin{cases} u = u_{\text{in}} + Qy \\ y = y_{\text{in}} + Pu \end{cases}$$

Figure 1: Controller Q for the transfer function P

stabilizing controller $= \left[\begin{smallmatrix} u_{\text{in}} \\ y_{\text{in}} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ is stable (H^∞).

A proper $\mathcal{B}(\mathtt{Y},\mathtt{U})$-valued function Q is called a (dynamic output feedback) **proper stabilizing controller** for a proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function P if the "input-to-error" map $E : \left[\begin{smallmatrix} u_{\text{in}} \\ y_{\text{in}} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in Figure 1 is stable $(E \in \mathrm{H}^\infty)$.The map E is obviously given by

$$E := \begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-QP)^{-1} & Q(I-PQ)^{-1} \\ P(I-QP)^{-1} & (I-PQ)^{-1} \end{bmatrix}. \tag{1}$$

(Observe that then P is also a proper stabilizing controller for Q.)

Right coprime

The following are equivalent for a proper holomorphic function P:

(i) P has a proper **stabilizing controller** Q (i.e., $\begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} \in \mathrm{H}^\infty$).

(ii) P has a right **coprime factorization**.

(iii) P has a **stabilizable and detectable realization**.

Two functions $M, N \in \mathrm{H}^\infty$ are called (Bézout) **right coprime** if $\begin{bmatrix} M \\ N \end{bmatrix}$ is left-invertible in H^∞, i.e., if there exist $\tilde{X}, \tilde{Y} \in \mathrm{H}^\infty$ satisfying the *Bézout identity*

$$\tilde{X}M - \tilde{Y}N \equiv I \qquad \text{(on } \mathbb{D}) . \tag{2}$$

We call the factorization $P = NM^{-1}$ a **right coprime factorization** of P if $N \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{Y})$ and $M \in \mathrm{H}^\infty(\mathtt{U})$ are right coprime, $M(0)$ is invertible and $P = NM^{-1}$.

Then $Q = \tilde{X}^{-1}\tilde{Y}$ is a stabilizing controller for P (if $\tilde{X}^{-1}$ exists).

All stabilizing controllers

Let P be $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued and have a right coprime factorization $P = NM^{-1}$. Then $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] \in \mathrm{H}^\infty(\mathtt{U},\mathtt{U}\times\mathtt{Y})$ can be extended to an invertible element of $\mathrm{H}^\infty(\mathtt{U}\times\mathtt{Y})$, say $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$. (This is called a *doubly coprime factorization* of P.) [Tolokonnikov81] [Treil04] [M05d]

All stabilizing controllers for P are given by the **Youla(–Bongiorno) parameterization**

$$Q = (Y + MV)(X + NV)^{-1} \tag{3}$$

where $V \in \mathrm{H}^\infty(\mathtt{Y},\mathtt{U})$ is arbitrary (the controller is proper iff $(X+NV)^{-1}$ is proper). [CuWeWe01] [M05d]

If P is strictly proper ($P(0) = 0$), then all these controllers are proper.

Matrix-valued case

Let P be a proper $\mathbb{C}^{n\times m}$-valued function. Then also the following are equivalent to the existence of a proper stabilizing controller:

(i*) P has a **stable** $(Q \in \mathrm{H}^\infty(\mathbb{C}^n, \mathbb{C}^m))$ stabilizing controller. [Treil92] [Quadrat04]

(ii*) $P = NM^{-1}$, where $N, M \in \mathrm{H}^\infty$, $N^*N + M^*M \geq \varepsilon I$ on $\mathbb{D}$, $\varepsilon > 0$ and $\det M \not\equiv 0$. [Carleson62] [Fuhrman68]

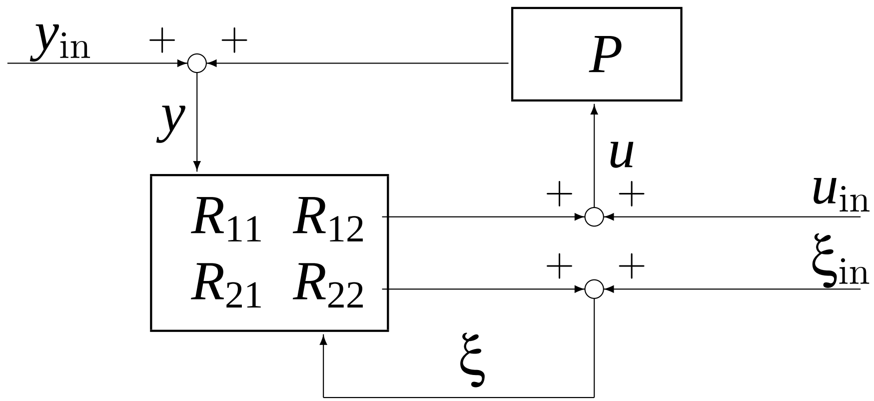
(The corona condition in (ii') is not sufficient for coprimeness in the operator-valued case [Treil89]. It is not known whether (i') is necessary in general.)

Controllers with internal loop

Also the following is equivalent to the existence of a proper stabilizing controller of P:

(i'') P has a stabilizing controller with internal loop. [CuWeWe01] [M05d]

We call R a **stabilizing controller with internal loop** for P if $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$ is a proper $\mathcal{B}(\mathtt{Y} \times \Xi, \mathtt{U} \times \Xi)$-valued function for some Hilbert space Ξ and the combined map $\begin{bmatrix} u_{\text{in}} \\ y_{\text{in}} \\ \xi_{\text{in}} \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \\ \xi \end{bmatrix}$ in Figure 2 becomes stable (H^∞).



$$y = y_{\text{in}} + Pu$$
$$\begin{bmatrix} u \\ \xi \end{bmatrix} = \begin{bmatrix} u_{\text{in}} \\ \xi_{\text{in}} \end{bmatrix} + R \begin{bmatrix} y \\ \xi \end{bmatrix}$$

Figure 2: Controller R with internal loop for P

If $I - R_{22}(0)$ is invertible, then R corresponds to the proper stabilizing controller $Q = R_{11} + R_{12}(I - R_{22})^{-1} R_{21}$.

Main Theorem (ver. 3)

The following are equivalent for a proper function P:

(i) P has a proper **stabilizing controller** Q (i.e., $\begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} \in \mathrm{H}^\infty$).

(i') P has a strictly proper stabilizing controller.

(i") P has a stabilizing controller with internal loop.

(ii) P has a **right coprime** factorization $P = NM^{-1}$.

(ii') P has a left coprime factorization $P = \tilde{M}^{-1}\tilde{N}$.

(ii") P has a doubly coprime factorization $P = NM^{-1}$, $\begin{bmatrix} M & Y \\ N & X \end{bmatrix}, \begin{bmatrix} M & Y \\ N & X \end{bmatrix}^{-1} \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$.

(iii) P has a **stabilizable and detectable realization**.

Discrete-time system $\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$

Given **input** $u \in \ell^2(\mathbb{N};\mathtt{U})$ and **initial state** $x_0 \in \mathtt{X}$, we associate the **state** trajectory $x : \mathbb{N} \to \mathtt{X}$ and **output** $y : \mathbb{N} \to \mathtt{Y}$ through

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k, \end{cases} \quad k \in \mathbb{N}. \tag{4}$$

The **transfer function** $P(z) := D + C(z^{-1} - A)^{-1}B$ of $\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ is proper.

We call $\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ a **realization** of P.

The **Z-transform** $\widehat{u}$ of $u : \mathbb{N} \to \mathtt{U}$ is defined by $\widehat{u}(z) := \sum_n z^n u_n$.

For $x_0 = 0$, we have $\widehat{y} = P\widehat{u}$.

State feedback $u_k = Fx_k$

State feedback means that we feed the state back to the input through some **state-feedback operator** $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$:

$$u_k := Fx_k + (u_{\text{in}})_k \qquad (k \in \mathbb{N}), \tag{5}$$

where u_{in} denotes an exogenous input (or disturbance), as in Figure 3.

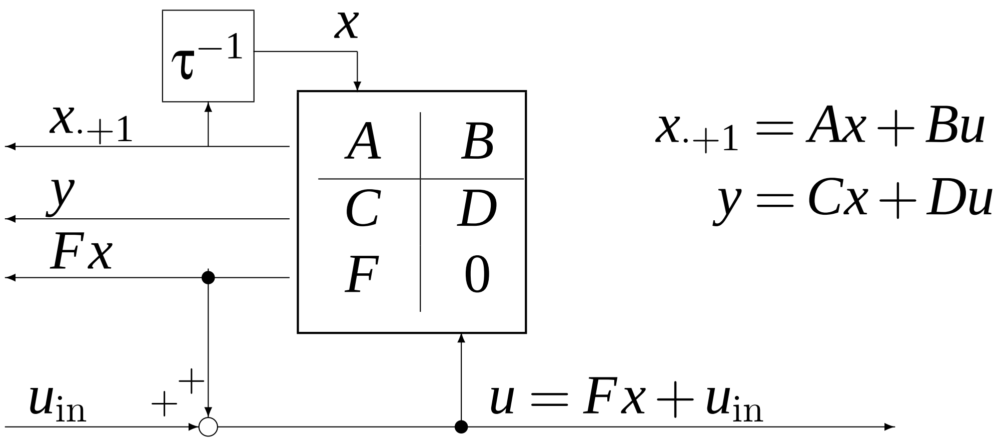


Figure 3: State-feedback connection

$$\Rightarrow \; x_{k+1} = (A+BF)x_k + B(u_{\text{in}})_k \; \Rightarrow \; \left(\begin{array}{c|c} A+BF & B \\ C & D \\ F & I \end{array}\right) : \begin{bmatrix} x_k \\ (u_{\text{in}})_k \end{bmatrix} \mapsto \begin{bmatrix} x_{k+1} \\ y_k \\ u_k \end{bmatrix}.$$

Closed-loop system

$$\left(\begin{array}{c|c} A+BF & B \\ \hline \begin{bmatrix} C+DF \\ F \end{bmatrix} & \begin{bmatrix} D \\ I \end{bmatrix} \end{array}\right) : \begin{bmatrix} x_k \\ (u_{\text{in}})_k \end{bmatrix} \mapsto \begin{bmatrix} x_{k+1} \\ y_k \\ u_k \end{bmatrix}. \tag{6}$$

The transfer function of the **closed-loop system** (6) is obviously given by

$$\begin{bmatrix} N(z) \\ M(z) \end{bmatrix} = \begin{bmatrix} D \\ I \end{bmatrix} + \begin{bmatrix} C+DF \\ F \end{bmatrix} (z^{-1} - A - BF)^{-1} B. \tag{7}$$

Because $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ maps $\widehat{u_{\text{in}}} \mapsto \begin{bmatrix} \widehat{y} \\ \widehat{u} \end{bmatrix}$, a factorization of $P : \widehat{u} \mapsto \widehat{y}$ is given by $P = NM^{-1}$.

Finite Cost Condition (FCC): For each $x_0 \in \mathtt{X}$, some $u \in \ell^2$ makes $y \in \ell^2$.

If(f) the FCC holds, then there exists $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ that minimizes $\sum_{k=0}^{\infty} (\|y_k\|_{\mathtt{Y}}^2 + \|u_k\|_{\mathtt{U}}^2)$ (LQR cost) for every x_0.

The resulting factorization $P = NM^{-1}$ is **weakly coprime** [M05a].
If the FCC holds for $\left(\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right)$, then $P = NM^{-1}$ is **right coprime** [CO05].

State-feedback stabilization of $\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$

$\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ **output stable** $= y \in \ell^2$ whenever $x_0 \in \mathtt{X}$ and $u = 0$;
i.e., $\|CA^{\cdot}x_0\|_2 \le K\|x_0\|_{\mathtt{X}}$ $(x_0 \in \mathtt{X})$.

$\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ **stable** $= y \in \ell^2$ and x is bounded whenever $x_0 \in \mathtt{X}$ and $u \in \ell^2(\mathbb{N};\mathtt{U})$; i.e.,

$$\|x_n\|_{\mathtt{X}} + \|y\|_2 \le K\left(\|x_0\|_{\mathtt{X}} + \|u\|_2\right) \qquad (n \ge 0,\ x_0 \in \mathtt{X},\ u \in \ell^2(\mathbb{N};\mathtt{U})). \tag{8}$$

$\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ **[output-]stabilizable** $= \left(\begin{array}{c|c} A+BF & B \\ C & D \\ F & I \end{array}\right)$ [output-]stable for some F.

$\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ **[input-]detectable** $= \left(\begin{array}{c|c} A^* & C^* \\ B^* & D^* \end{array}\right)$ [output-]stabilizable.

(iii) P has a **stabilizable and detectable realization**.

(iii') P has an **output-stabilizable and input-detectable realization**.

Theorem Output-stabilizability⇔Finite Cost Condition. [M05a]

(iii") P has a realization $\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ such that $\left(\begin{array}{c|c} A & B \\ C & D \end{array}\right)$ and $\left(\begin{array}{c|c} A^* & C^* \\ B^* & D^* \end{array}\right)$ satisfy the Finite Cost Condition.

Main theorem (ver. 4)

The following are equivalent for a proper function P:

(i) P has a proper **stabilizing controller** Q (i.e., $\begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} \in \mathrm{H}^\infty$).

(i''') P has a realization that has a **stabilizing controller system**.

(ii) P has a **right coprime** factorization $P = NM^{-1}$.

(iii) P has a **stabilizable and detectable realization**.

(iii') P has an output-stabilizable and input-detectable realization.

(iii'') P has a realization $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ such that $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ and $\left(\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right)$ satisfy the Finite Cost Condition.

(iii''') P has a strongly stabilizable and strongly detectable realization.
("Strongly" means that, in addition, $x_k \to 0$, as $k \to +\infty$.)

Dynamic output-feedback stabilization

(i''') P has a realization that has a **stabilizing controller system** $\left(\begin{array}{c|c}\tilde A & \tilde B \\ \tilde C & \tilde D\end{array}\right)$.

This says that if we feed the output of $\left(\begin{array}{c|c}A & B \\ C & D\end{array}\right)$ through $\left(\begin{array}{c|c}\tilde A & \tilde B \\ \tilde C & \tilde D\end{array}\right)$ back to the input, then the combined system becomes stable, i.e., in Figure 4

$$\left\|\left[\begin{smallmatrix}x_n\\ \tilde x_n\end{smallmatrix}\right]\right\|_{\mathtt{X}\times\tilde{\mathtt{X}}}+\left\|\left[\begin{smallmatrix}y\\ u\end{smallmatrix}\right]\right\|_2\le K\left(\left\|\left[\begin{smallmatrix}x_0\\ \tilde x_0\end{smallmatrix}\right]\right\|_{\mathtt{X}\times\tilde{\mathtt{X}}}+\left\|\left[\begin{smallmatrix}y_{\rm in}\\ u_{\rm in}\end{smallmatrix}\right]\right\|_2\right)\qquad \left(n\ge 0,\ \left[\begin{smallmatrix}x_0\\ \tilde x_0\end{smallmatrix}\right]\in\mathtt{X}\times\tilde{\mathtt{X}},\ \left[\begin{smallmatrix}y_{\rm in}\\ u_{\rm in}\end{smallmatrix}\right]\in\ell^2(\mathbb{N};\mathtt{Y}\times\mathtt{U})\right).$$

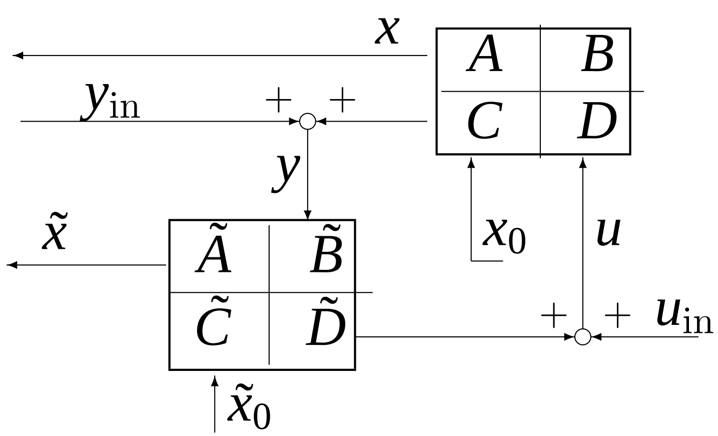


Figure 4: Stabilizing controller system

This implies that $Q(z)=\tilde D+\tilde C(z^{-1}-\tilde A)^{-1}\tilde B$ is a proper stabilizing controller for $P(z)=D+C(z^{-1}-A)^{-1}B$ (i.e., $\left[\begin{smallmatrix}I & -Q\\ -P & I\end{smallmatrix}\right]^{-1}\in \mathrm{H}^\infty$).

The converse holds iff $\left(\begin{array}{c|c}A & B \\ C & D\end{array}\right)$ and $\left(\begin{array}{c|c}\tilde A & \tilde B \\ \tilde C & \tilde D\end{array}\right)$ are stabilizable and detectable [M05e].

Main theorem (final version)

The following are equivalent for a proper function P:

(i) P has a proper **stabilizing controller** Q (i.e., $\begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} \in \mathrm{H}^\infty$).

(i") P has a stabilizing controller with internal loop. [CuWeWe01] [M05d]

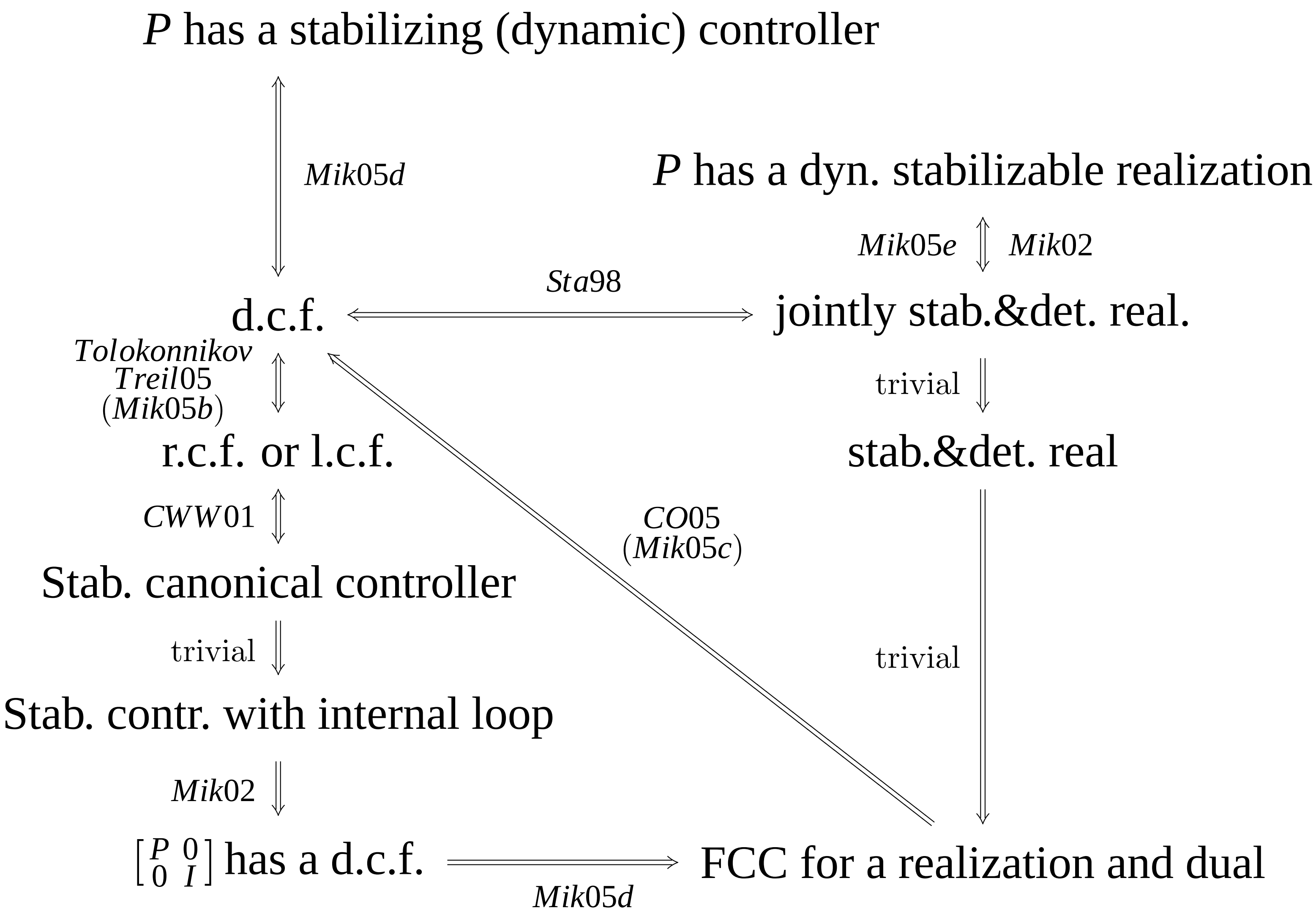
(i"') P has a realization that has a **stabilizing controller system** $\left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right)$. [M05e]

(ii) P has a **right coprime** factorization $P = NM^{-1}$. [Smith] [M05d]

(ii") P has a doubly coprime factorization $P = NM^{-1}$, $\begin{bmatrix} M & Y \\ N & X \end{bmatrix}, \begin{bmatrix} M & Y \\ N & X \end{bmatrix}^{-1} \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$.

(iii) P has a **stabilizable and detectable realization**. [St98] [CO05] [M05c]

Proof of part of the Main Theorem



Weaker equivalent conditions

(i) P has a proper stabilizing controller Q (i.e., $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$).

(ii) P has a right coprime factorization $P = NM^{-1}$.

(ii") P has a doubly coprime factorization $P = NM^{-1}$, $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right], \left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$.

(iii) P has a stabilizable and detectable realization.

(iii') P has an output-stabilizable realization whose dual is output-stabilizable.

The following (strictly weaker) conditions equivalent to each other:

(ii-) P has a **factorization $P = NM^{-1}$** ($N, M \in \mathrm{H}^\infty$).

(ii'-) P has a **weakly coprime** factorization $P = NM^{-1}$. [M05a]

(iii-) P has a **stabilizable realization**.

(iii'-) P has an output-stabilizable realization. [M02]

(i-) The range of the generalized Hankel operator of P lies in the range of the generalized Toeplitz operator of P plus H^2. [M05c]

Weakly coprime = common right factors are units

Scalar-valued case ($\mathtt{U}=\mathbb{C}=\mathtt{Y}$):
$N \in \mathrm{H}^\infty(\mathtt{U},\mathtt{Y})$ and $M \in \mathrm{H}^\infty(\mathtt{U},\mathtt{U})$ are weakly coprime iff $\gcd(N,M)=I$.

Equivalent condition: if $N=N_1V$ and $M=M_1V$, then V is a unit ($V,V^{-1}\in\mathrm{H}^\infty$). [Fuhrmann81] [Smith89]

Equivalent condition:
if $N=N_1V$ and $M=M_1V$ and $V(0)$ is invertible, then V is a unit ($V,V^{-1}\in\mathrm{H}^\infty$).

I.e., every properly-invertible common right factor is a unit.

This latter condition is meaningful also when $\mathtt{U}$ and $\mathtt{Y}$ are infinite-dimensional (the former is then never satisfied). An equivalent condition is

$$Nf, Mf \in \mathrm{H}^2 \implies f \in \mathrm{H}^2 \tag{9}$$

(for every proper $\mathtt{U}$-valued function f). Either of these two conditions can be used as the definition of **weak right coprimeness**.

One obtains a third equivalent condition by replacing H^2 by H^∞ in (9).

Generalized Toeplitz and Hankel ranges

(ii-) P has a **factorization** $P = NM^{-1}$ $(N, M \in \mathrm{H}^\infty)$.

(ii'-) P has a **weakly coprime** factorization $P = NM^{-1}$.

(iii-) P has a **stabilizable realization**.

(i-) $\mathrm{Ran}(H_P) \subset \mathrm{Ran}(T_P) + \mathrm{H}^2$.

(i'-) $\exists r > 1\ \forall v \in \ell^2_r(\mathbb{Z}_-; \mathtt{U})\ \exists u \in \ell^2(\mathbb{N}; \mathtt{U})$ such that $\mathscr{D}(v+u) \in \ell^2(\mathbb{N}; \mathtt{Y})$

T_P is the "unbounded Toeplitz operator" that maps $\mathrm{H}^2 \ni \widehat{u} \mapsto P\widehat{u}$

H_P is the "unbounded Hankel operator" that maps
$\mathrm{H}^2(r\mathbb{D}^-; \mathtt{U}) \ni \widehat{v} \mapsto$ projection of $P\widehat{v}$ onto $\mathrm{H}^2_r := \mathrm{H}^2(r\mathbb{D}; \mathtt{Y})$ (for some big r).

The *I/O map* $\mathscr{D}$ is determined by $\widehat{\mathscr{D}u} = P\widehat{u}$. It has a unique continuous extension to a map $\mathscr{D} : \ell^2_r \to \ell^2_r$ for every big r, where $\|u\|_{\ell^2_r}{}^2 := \sum_{k=-\infty}^{\infty} r^{2k} \|u_k\|^2_{\mathtt{U}}$.

Note that $T_P\widehat{u} = \widehat{\ }(\pi_+ \mathscr{D} \pi_+)$ and $H_P\widehat{u} = \widehat{\ }(\pi_+ \mathscr{D} \pi_-)$,
where $(\pi_+ u)_k := \begin{cases} u_k, & k \geq 0; \\ 0, & k < 0 \end{cases}$, $\pi_- := I - \pi_+$. We have set $r\mathbb{D}^- := \{z \in \mathbb{C} \mid |z| > r\}$.

Naturally, $\|\widehat{u}\|_{\mathrm{H}^2_r}{}^2 := \|\widehat{u}(r\cdot)\|^2_{\mathrm{H}^2} = \sup_{t<r} \int_0^{2\pi} \|\widehat{u}(te^{i\theta})\|^2_{\mathtt{U}}\, d\theta = 2\pi \|\widehat{u}\|^2_{\ell^2_r}$.

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