COPRIME FACTORIZATION AND DYNAMIC STABILIZATION OF TRANSFER FUNCTIONS

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Abstract. It is known that a matrix-valued transfer function P has a stabilizing dynamic controller Q (i.e., $\begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} \in \mathrm{H}^\infty$) iff P has a right (or left) coprime factorization. We show that the same result is true in the operator-valued case. Thus, the standard Youla–Bongiorno parameterization applies to every dynamically stabilizable function. We then derive further equivalent conditions, one of them being that P has a stabilizing controller with internal loop; this and some others are new even in the scalar-valued case.

We also establish certain related results. For example, we extend the classical results on coprime factorization and partial feedback (measurement-feedback) stabilization to nonrational transfer functions.

All our results apply in both discrete- and continuous-time settings, except that in the latter it is not clear whether the controller Q can always be chosen so that it is "continuous-time proper" (holomorphic and bounded on a right half-plane) unless, e.g., $P(z) \to 0$ as $\operatorname{Re} z \to +\infty$.

Key words. dynamic stabilization, internal stabilization, right coprime factorization, measurement feedback, dynamic partial feedback, dynamically stabilizing controllers with internal loop, operator-valued transfer functions, infinite-dimensional systems

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1. Introduction.

In this introductory section we present our main results for discrete-time transfer functions (those defined on a subset of the unit disc $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$). Corresponding results for continuous-time functions (those defined on the right half-plane) and others are given in §7.

Let $\mathtt{U}$, $\mathtt{W}$, $\mathtt{Y}$ and $\mathtt{Z}$ be complex Hilbert spaces. By $\mathcal{B}(\mathtt{U},\mathtt{Y})$ we denote bounded linear operators $\mathtt{U} \to \mathtt{Y}$ and by $\mathrm{H}^\infty(\mathtt{U},\mathtt{Y})$ we denote bounded holomorphic functions $\mathbb{D} \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ with supremum norm. We set $\mathcal{B}(\mathtt{U}) := \mathcal{B}(\mathtt{U},\mathtt{U})$, $\mathrm{H}^\infty(\mathtt{U}) := \mathrm{H}^\infty(\mathtt{U},\mathtt{U})$, $\mathcal{GB} := \{F \in \mathcal{B} \mid \text{there exists } F^{-1} \in \mathcal{B}\}$ and $\mathcal{G}\,\mathrm{H}^\infty := \{F \in \mathrm{H}^\infty \mid \text{there exists } F^{-1} \in \mathrm{H}^\infty\}$. By I or $I_{\mathtt{U}}$ we denote the identity operator $I \in \mathcal{B}(\mathtt{U})$ (or the corresponding constant function $I \in \mathrm{H}^\infty(\mathtt{U})$).

A holomorphic function P ("the plant") defined on a neighborhood of the origin is called *proper*. It is *strictly proper* if $P(0) = 0$. We identify a holomorphic function on a disc $r\mathbb{D} = \{z \in \mathbb{C} \mid |z| < r\}$ with its restriction to any open subset of $r\mathbb{D}$.

A proper $\mathcal{B}(\mathtt{Y},\mathtt{U})$-valued function Q is called a (dynamic feedback) *proper stabilizing controller* for a proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function P if the "input-to-error" map $E : \left[\begin{smallmatrix} u_{\text{in}} \\ y_{\text{in}} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in Figure 1.1 is in H^∞.[1] The map E is obviously given by

$$E := \begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-QP)^{-1} & Q(I-PQ)^{-1} \\ P(I-QP)^{-1} & (I-PQ)^{-1} \end{bmatrix}. \tag{1.1}$$

(Observe that then P is also a proper stabilizing controller for Q.)

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[1] This means that some $E \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$ satisfies $E \left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right] E$ on a neighborhood of 0. By a direct computation, (1.1) follows (on a neighborhood of 0). Recall that the inverse of a holomorphic operator-valued function is always holomorphic. (This kind of algebraic, function-theoretic and other well-known results used in this article can be found in our generality in [11, Appendices A & D].)

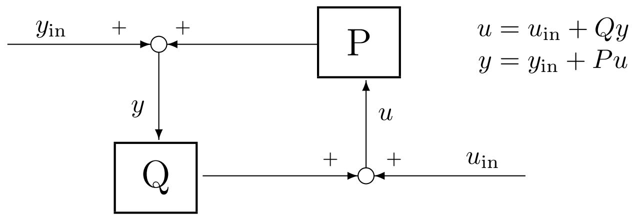

FIG. 1.1. *Controller Q for the transfer function P*

Two functions $M, N \in \mathrm{H}^\infty$ are called (Bézout) *r.c.* (right coprime) if $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ is left-invertible in H^∞, i.e., if there exist $\tilde{X}, \tilde{Y} \in \mathrm{H}^\infty$ satisfying the "Bézout identity"

$$\tilde{X}M - \tilde{Y}N \equiv I. \tag{1.2}$$

We call the factorization $P = NM^{-1}$ a *r.c.f.* (right coprime factorization) of P if $N \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{Y})$ and $M \in \mathrm{H}^\infty(\mathtt{U})$ are r.c., $M(0) \in \mathcal{GB}$ and $P = NM^{-1}$ (near 0).

The following is our main result:

THEOREM 1.1 (Dynamic feedback stabilization). *The following are equivalent for any proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function P:*

(i) *P has a strictly proper stabilizing controller.*
(ii) *P has a proper stabilizing controller.*
(iii) *P has a stabilizing controller with internal loop.*[2]
(iv) *P has a r.c.f.*
(v) *$\left[\begin{smallmatrix} P & 0 \\ 0 & I_{\mathtt{Z}} \end{smallmatrix}\right]$ has a r.c.f. for some (hence any) Hilbert space $\mathtt{Z}$.*

Assume that P has a r.c.f. $P = NM^{-1}$. Then $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{U} \times \mathtt{Y})$ can be extended to an invertible element of $\mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$, say $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$. Denote its inverse by $\left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right] \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$. Then all stabilizing controllers for P are given by the Youla(–Bongiorno) parameterization[3]

$$Q = (Y + MV)(X + NV)^{-1} \qquad (= (\tilde{X} + V\tilde{N})^{-1}(\tilde{Y} + V\tilde{M})), \tag{1.3}$$

where $V \in \mathrm{H}^\infty(\mathtt{Y}, \mathtt{U})$ is arbitrary (the controller is proper iff $(X + NV)^{-1}$ is proper, or equivalently, iff $(\tilde{X} + V\tilde{N})^{-1}$ is proper). The map $V \mapsto Q$ is one-to-one.

If P is strictly proper, then all these controllers are proper.

Usually one excludes the values of the parameter V that make the controller (1.3) non-proper. However, sometimes only such controllers possess the properties that one would like to obtain in practical applications [4]. To include also such controllers, the theory of "controllers with internal loop" (which cover both the proper and non-proper controllers) was developed in [32] and [4]. Also non-proper controllers with internal loop can be physically realized. In §3 we shall define them and explain their relation to proper controllers.

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[2]These will be defined in Section 3. They may be non-proper.

[3]For some functions P and V, the inverse $(X + NV)^{-1}$ in (1.3) need not exist at the origin (or anywhere; e.g., $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $V = 0$). Even so, the "non-proper" controller (1.3) can be interpreted as a "stabilizing controller with internal loop", as described in §3, where also properness is explained in detail. Nevertheless, for each P (and $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$), some $V \in \mathrm{H}^\infty$ makes $(X + NV)(0)$ invertible in $\mathcal{B}(\mathtt{Y})$. The parameterization (1.3) covers all stabilizing controllers with internal loop in the sense described in §3. Moreover, every proper stabilizing controller equals exactly one of these Q on a neighborhood of the origin.

Further necessary and sufficient conditions for (i) will be presented later, particularly in Theorem 2.1 and Proposition 2.2. One such condition is the existence of a stabilizable and detectable realization. Conditions (iii) and Theorem 2.1(ii') are weaker forms of (ii). Their equivalence to (ii) means that if P is dynamically stabilizable in any reasonable sense, then it is dynamically stabilizable in the standard sense (possibly by a different, nonequivalent controller).

By combining the above results with [27], [29] and [7], we obtain the following result:

COROLLARY 1.2 (Matrix-valued case). *Assume that* $\dim \mathtt{U} < \infty$ *and* $\dim \mathtt{Y} < \infty$. *Then also the following conditions are equivalent to (i) of Theorem 1.1 for a proper* $\mathcal{B}(\mathtt{U}, \mathtt{Y})$*-valued function* P*:*

(vi) *P has a stable (hence proper) stabilizing controller ($Q \in \mathrm{H}^\infty(\mathtt{Y}, \mathtt{U})$).*

(vii) $P = NM^{-1}$*, where* $N, M \in \mathrm{H}^\infty$*,* $N^*N + M^*M \geq \epsilon I$ *on* $\mathbb{D}$*,* $\epsilon > 0$ *and* $\det M \not\equiv 0$.

(The corona condition in (vii) is not sufficient for coprimeness in the operator-valued case [26]. It is not known whether (vi) is necessary in general.)

For rational transfer functions, (i)–(vii) always hold and also the rest of Theorem 1.1 is well known [6]. The study of corresponding results for nonrational functions started in the 1970s and soon became intensive. An introduction to coprime factorization and dynamic stabilization of infinite-dimensional systems can be found in, e.g., [5] or [31]. Several sufficient conditions for some of the conditions (i)–(v) have been established earlier, but our proof of the equivalence would not have been possible without the results in [23], [32], [11], [14], [2] and [16].

In the matrix-valued case, the implication (ii)$\Rightarrow$(iv) was independently established in [9] and [22] and the enhanced converse (iv)$\Rightarrow$(vi) in [27] (in the scalar-valued case, which is equivalent to the matrix-valued case, by [29, Theorem 3]; the exact statement can be found in [19]). The "Carleson Corona Theorem" (iv)$\Leftrightarrow$(vii) was extended to the matrix-valued case in [7] (see [28] for the operator-valued case, where (vii)$\Rightarrow$(iv) is not true without additional assumptions).

In the general case, the implication (iv)$\Rightarrow$(iii) was established in [32] and [4], and (iv)$\Rightarrow$(v) and (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are trivial. The existence of $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$ is from [16] (based on [28] and [12]); the matrix-valued case is a well-known consequence of Tolokonnikov's Lemma [25]. The Youla parameterization (including the "if" part of the properness of Q) is straightforward [4]. The "only if" part of properness, implications (iii)$\Rightarrow$(v)$\Rightarrow$(iv)$\Rightarrow$(i) and the strictly proper case are new (except that (iv)$\Rightarrow$(i) was already known in the matrix-valued case). The differences between continuous- and discrete-time results are otherwise insignificant, but properness and strict properness become more complicated in continuous time; see §7 for details.

With certain other commutative unital rings in place of $\mathrm{H}^\infty(\mathbb{C})$, Theorem 1.1 becomes false. Related results for such settings are given by Alban Quadrat [19] [18] [20], in the matrix-valued case.

In §2 we present further conditions that are equivalent to (i), such as coprime factorization or stabilization with invertibility at some other $\alpha \in \mathbb{D}$ instead of 0. In §3 we define controllers with internal loop, present corresponding details of Theorem 1.1 and develop related new results. The results in §2 and §3 are needed in the proof of Theorem 1.1 but they are also important by themselves.

In §4 we present analogous results for "measurement feedback" or dynamic partial feedback, where the controller can use only a part of the output and can affect only a part of the input of $P = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]$, where $P(z) \in \mathcal{B}(\mathtt{U} \times \mathtt{W}, \mathtt{Z} \times \mathtt{Y})$, as in Figure 1.2. We obtain direct generalizations of the classical results, such as those in [6] or [8].

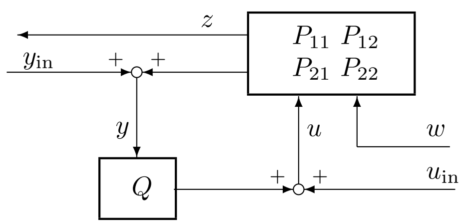


FIG. 1.2. *DPF-controller Q for P*

In particular, we show that if P is stabilizable by dynamic partial feedback, then a $\mathcal{B}(\mathtt{Y},\mathtt{U})$-valued controller Q stabilizes P by dynamic partial feedback iff it stabilizes P_{21} by dynamic feedback.

In §5 we observe that practically all our results also hold for "power stabilization" (or "exponential stabilization" in the continuous-time setting of §7), mutatis mutandis, where the "closed-loop" map (1.1) is required to be holomorphic on an open set that contains $\overline{\mathbb{D}}$. In §6 we show that even if we allow the domain of Q to be an arbitrary region, we meet no ambiguity with holomorphic extensions and the identification of controllers.

In §7 we establish our results in the continuous-time setting, where the properness notion is different. Proofs and some further results are given in the appendices.

In our generality, corresponding state-space results can be found in [33], [11] and [24] (and [32]), where many assumptions can be weakened, by our results. Robust stabilization with state-space results are given in [1]. Further state-space results will be presented in a subsequent article by the author.

2. Dynamic stabilization. In this section we show how any reasonable variants of the above conditions (i)–(v) are equivalent to (i). We also present realization-based conditions that are equivalent to (i).

In the matrix-valued case, as in (vii), one need not care where M or $\begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1}$ is invertible, since it is invertible a.e. anyway (if it is invertible somewhere). In Theorem 2.1 we show that invertibility at any reasonable point is sufficient also in the operator-valued case. The definitions below are used to formulate these facts.

Let $\alpha \in \mathbb{D}$. We call NM^{-1} an *α-r.c.f. of P* if $N, M \in \mathrm{H}^\infty$ are r.c., $M(\alpha) \in \mathcal{GB}(\mathtt{U})$ and $NM^{-1} = P$ on a neighborhood of α. We call $\tilde{M}^{-1}\tilde{N}$ an *α-l.c.f. of P* if $\tilde{N}, \tilde{M} \in \mathrm{H}^\infty$ are *l.c.* (i.e., $\tilde{M}X - \tilde{N}Y = I$ for some $X, Y \in \mathrm{H}^\infty$), $\tilde{M}(\alpha) \in \mathcal{GB}(\mathtt{U})$ and $\tilde{M}^{-1}\tilde{N} = P$ on a neighborhood of α. We call $\begin{bmatrix} M & Y \\ N & X \end{bmatrix} \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U}\times\mathtt{Y})$ an *α-d.c.f. of P* if $M(\alpha) \in \mathcal{GB}(\mathtt{U})$ and $P = NM^{-1}$ on a neighborhood of α (it follows that $P = NM^{-1}$ is an α-r.c.f. and $P = \tilde{M}^{-1}\tilde{N}$ is an α-l.c.f., where $\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} := \begin{bmatrix} M & Y \\ N & X \end{bmatrix}^{-1}$ [11]; conversely, any α-r.c.f. and α-l.c.f. can be extended to an α-d.c.f., by Lemma 3.4). A 0-d.c.f. (resp., 0-l.c.f.) is called a *d.c.f.* (resp., *l.c.f.*).

Now we can present further equivalent conditions (see §3 for (iii')):

THEOREM 2.1 (Dynamic feedback stabilization). *Assume that $\Omega_P \subset \mathbb{D}$ is open and connected, $P : \Omega_P \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ is holomorphic and $0, \alpha, \beta \in \Omega_P$. Then the following conditions are equivalent to (iv) of Theorem 1.1:*

(iv') *P has an α-r.c.f.*

(iv") *P has an α-l.c.f.*

(iv"') *P has an α-d.c.f.*

(ii') *For some open and connected $\Omega_Q \subset \Omega_P$ there exists a holomorphic function*

$Q : \Omega_Q \to \mathcal{B}(\mathtt{Y}, \mathtt{U})$ such that $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$.

(ii") *For some neighborhood Ω_Q of α, condition (ii') holds with $Q(\alpha) = 0$.*

(iii') *P has a stabilizing canonical controller.*

Any α-r.c.f. of P is a β-r.c.f. of P. The same holds with "l.c.f." or "d.c.f." in place of "r.c.f.".

Note that, by duality, we get "left results" from all "right results" of this article (because, e.g., $P = \tilde{M}^{-1}\tilde{N}$ is an α-l.c.f. iff $P^{\mathrm{d}} = \tilde{N}^{\mathrm{d}}(\tilde{M}^{\mathrm{d}})^{-1}$ is an $\bar{\alpha}$-r.c.f., where the *dual* P^{d} is defined by $P^{\mathrm{d}}(s) := P(\bar{s})^*$). See §6 for further variants of (ii').

We recall the following from [16] (which contains the definitions of (viii)–(viii"')):

Proposition 2.2 (Realizations). *Assume that P is a proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function. Then also the following are equivalent to (i) of Theorem 1.1:*

(viii) *P has a jointly stabilizable and detectable realization.*

(viii') *P has a stabilizable and detectable realization.*

(viii") *P has an output-stabilizable and input-detectable realization.*

(viii"') *P has a realization Σ such that Σ and its dual satisfy the Finite Cost Condition.*

See [2] or [16] for an equivalent condition in terms of Riccati equations, which also yield a constructive formula for the r.c.f. The original proof of "(viii')$\Rightarrow$(iv)" is due to [2], and that of "(viii)$\Leftrightarrow$(iv"')" due to [23], both in continuous time.

By combining Proposition 2.2 and Theorem 1.1 with [11] one can obtain further equivalent conditions, such as having a dynamically stabilizable realization. For (continuous-time) exponential dynamic stabilization of realizations, the necessity of exponential stabilizability and detectability was shown in [33]; their sufficiency follows from [17] (or [33]), Remark 5.1 and Theorems 1.1 and 7.3.

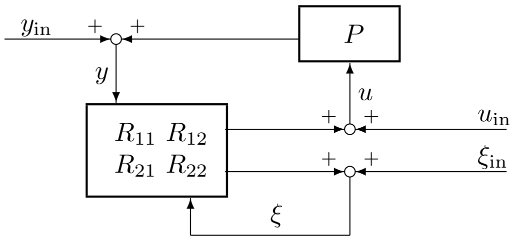
Constructive formulae for doubly coprime factorizations in terms of realizations can be found in, e.g., [2], [3] and [11] under different assumptions; in [32], [11] and [1] formulae for stabilizing dynamic controllers are given. They also provide further historical remarks. For constructive formulae for mere r.c.f.'s, see also the end of Section 7.

3. Controllers with internal loop. In this section we present certain results on controllers with internal loop and explain the rest of Theorem 1.1. As before, we work in the discrete-time setting but we show in §7 that practically everything below holds in the continuous-time setting too.

Controllers with internal loop were defined in [32] both to complete the theory of dynamic stabilization of nonrational transfer functions and to cover also the "short circuit control" type applications. Their theory has been further developed in [4], [33], and [11]. As explained in [32] and [4], without them some aspects of the standard theory for finite-dimensional systems cannot be satisfactorily generalized to general infinite-dimensional systems. E.g., the standard observer-based controller need not have a proper transfer function but it can be identified with a proper 2×2-matrix-valued transfer function [32, Example 6.5], which has a well-posed realization. See also the rational SISO example at the end of this section.

We start this section from the definitions and then explain the correspondence to the proper controllers presented in the introduction.

We say that R is a (possibly non-proper) *stabilizing controller with internal loop* for P if $R = \left[\begin{smallmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{smallmatrix}\right]$ is a proper $\mathcal{B}(\mathtt{Y} \times \Xi, \mathtt{U} \times \Xi)$-valued function for some Hilbert



$$y = y_{\text{in}} + Pu$$
$$\begin{bmatrix} u \\ \xi \end{bmatrix} = \begin{bmatrix} u_{\text{in}} \\ \xi_{\text{in}} \end{bmatrix} + R \begin{bmatrix} y \\ \xi \end{bmatrix}$$

FIG. 3.1. *Controller R with internal loop for P*

space Ξ and

$$(I - P^R)^{-1} \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y} \times \Xi), \quad \text{where} \quad P^R = \begin{bmatrix} 0 & R_{11} & R_{12} \\ P & 0 & 0 \\ 0 & R_{21} & R_{22} \end{bmatrix}. \tag{3.1}$$

Note that $(I - P^R)^{-1}$ maps $\left[\begin{smallmatrix} u_{\text{in}} \\ y_{\text{in}} \\ \xi_{\text{in}} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \\ \xi \end{smallmatrix}\right]$ in Figure 3.1. Thus, R is stabilizing iff the maps $\left[\begin{smallmatrix} u_{\text{in}} \\ y_{\text{in}} \\ \xi_{\text{in}} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \\ \xi \end{smallmatrix}\right]$ are "well-posed and stable".

If $R = \left[\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then R is completely equivalent to the stabilizing controller Q. Thus, the proper controllers presented in the introduction essentially form a subset of the controllers with internal loop.

In general, R corresponds to "$R_{11} + R_{12}(I - R_{22})^{-1}R_{21}$" (cf. Lemma 3.2); this "function" need not be proper (we may even have $R_{22} \equiv I$). In the non-proper case the ξ-loop in Figure 3.1 becomes ill-posed if R is disconnected from P, i.e., physically one must connect R to P before closing the internal loop.

A non-proper controller is a proper controller for an extended system:

LEMMA 3.1. *A proper $\mathcal{B}(\mathtt{Y} \times \Xi, \mathtt{U} \times \Xi)$-valued function R is a stabilizing controller with internal loop for P iff R is a stabilizing controller for $P_I := \left[\begin{smallmatrix} P & 0 \\ 0 & I_\Xi \end{smallmatrix}\right]$.*

(This follows because $\left[\begin{smallmatrix} I & -R \\ -P_I & I \end{smallmatrix}\right]^{-1}$ consists of $(I - P^R)^{-1}$ and of some copies of its elements, as observed in [11, Proposition 7.2.5(c)]. An alternative proof is to observe that the equations that determine the latter reduce to those that determine the former. In fact, this is rather obvious, since I_Ξ corresponds to the identity feedthrough of ξ in Figure 3.1.)

Two stabilizing controllers with internal loop are considered *equivalent* for P iff they lead to the same closed-loop map $\left[\begin{smallmatrix} u_{\text{in}} \\ y_{\text{in}} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ (i.e., if the (1–2, 1–2)-blocks of corresponding $(I - P^R)^{-1}$'s are equal), even if the maps from ξ_{in} and the maps to ξ (i.e., those describing the internal loop in the controller) would differ.

In the lemma below we show that R corresponds to a proper controller Q iff the internal loop of R can be closed (while R is disconnected from P):

LEMMA 3.2 (Proper R). *Assume that $R = \left[\begin{smallmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{smallmatrix}\right]$ is a stabilizing controller with internal loop for P.*

Then R is equivalent to a stabilizing controller with internal loop of form $\tilde{R} = \left[\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right]$ iff $I - R_{22}(0) \in \mathcal{GB}$. If $I - R_{22}(0) \in \mathcal{GB}$, then the unique solution is given by $Q = R_{11} + R_{12}(I - R_{22})^{-1}R_{21}$.

(Note that then $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$, as in the introduction, and that if we close the internal loop of R, then its top-left block becomes $R_{11} + R_{12}(I - R_{22})^{-1}R_{21}$.)

Any such R is called a *proper* stabilizing controller (with internal loop) for P, (and R is identified with Q). Since equivalence is an equivalence relation, R is equivalent to a proper stabilizing controller with internal loop iff R is proper, by Lemma 3.2.

If $Y \in \mathrm{H}^\infty(\mathtt{Y},\mathtt{U})$ and $X \in \mathrm{H}^\infty(\mathtt{U})$ are r.c., then $R := \left[\begin{smallmatrix} 0 & Y \\ I & I-X \end{smallmatrix}\right]$ is called a *canonical controller* (see [4] or [11]; in [11], the term controller with a coprime internal loop was used). Sometimes we denote it by YX^{-1}, as in the Youla parameterization (1.3) above. In particular, we say that YX^{-1} *stabilizes* P iff $\left[\begin{smallmatrix} 0 & Y \\ I & I-X \end{smallmatrix}\right]$ is a stabilizing controller with internal loop for P.

(If YX^{-1} is a stabilizing canonical controller for P, then it is equivalent to $\left[\begin{smallmatrix} 0 & I \\ \tilde{Y} & I-\tilde{X} \end{smallmatrix}\right]$ (or $\tilde{X}^{-1}\tilde{Y}$), where $\tilde{X}$ and $\tilde{Y}$ are obtained by $\left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right] := \left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]^{-1}$ for any r.c.f. NM^{-1} of P, as one observes from Lemma 3.5 and its dual.)

Modulo equivalence, there are no other controllers than the canonical ones:

LEMMA 3.3 (Equivalent canonical controller). *Let R be a stabilizing controller with internal loop for P. Then some stabilizing canonical controller $\tilde{X}^{-1}\tilde{Y}$ for P is equivalent to R (and so is one of form YX^{-1}).*

The Youla parameterization (1.3) gives all stabilizing canonical controllers for P. Here we have identified the canonical controllers that are equivalent (see above); i.e., the ones determined by $\left[\begin{smallmatrix} Y \\ X \end{smallmatrix}\right] V$ for a fixed r.c. pair $\left[\begin{smallmatrix} Y \\ X \end{smallmatrix}\right]$ and arbitrary $V \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U})$. Any stabilizing controller with internal loop is equivalent to exactly one stabilizing canonical controller, so actually all stabilizing controllers with internal loop are covered by (1.3) modulo equivalence. In particular, this parameterization contains all proper stabilizing controllers, by Lemma 3.2. (Indeed, any proper stabilizing controller is equivalent to one of form $\tilde{R} := \left[\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and to a canonical controller $R := \left[\begin{smallmatrix} 0 & Y \\ I & I-X \end{smallmatrix}\right]$. By Lemma 3.2, $X(0) = I - R_{22}(0) \in \mathcal{GB}$ and $Q = YX^{-1}$.)

Since any r.c.f. and l.c.f. can be extended to a d.c.f. (Lemma 3.4), we can apply (1.3) when we have either.

A "Bézout identity" $\tilde{X}M - \tilde{Y}N = I$ (or $\tilde{M}X - \tilde{N}Y = I$) can always be extended to a d.c.f.:

LEMMA 3.4 (r.c.f.→d.c.f.). *Let $M, N, \tilde{X}, \tilde{Y} \in \mathrm{H}^\infty$ be such that NM^{-1} is a $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued r.c.f. and $\tilde{X}M - \tilde{Y}N = I$. Then, for any l.c.f. $\tilde{M}^{-1}\tilde{N}$ of NM^{-1}, there exist $X, Y \in \mathrm{H}^\infty$ such that $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right]^{-1} \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U}\times\mathtt{Y})$.*

(This follows from the proof of [23, Lemma 4.3(iii)]; observe from Theorem 2.1 that the l.c.f. necessarily exists.)

A canonical controller YX^{-1} stabilizes P iff $\left[\begin{smallmatrix} Y \\ X \end{smallmatrix}\right]$ can be extended to a d.c.f. of P. We state the dual result here:

LEMMA 3.5. *Let P be a proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function and $R' = \left[\begin{smallmatrix} 0 & I \\ \tilde{Y} & I-\tilde{X} \end{smallmatrix}\right] \in \mathrm{H}^\infty(\mathtt{Y}\times\mathtt{U},\mathtt{U}\times\mathtt{U})$.*

Then R' is a stabilizing controller with internal loop for P iff for some (hence any) r.c.f. NM^{-1} of P we have $\tilde{X}M - \tilde{Y}N \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U})$, or equivalently, iff P has a r.c.f. NM^{-1} such that $\tilde{X}M - \tilde{Y}N = I$.

Assume that R' is a stabilizing controller with internal loop for P. Then there exists a d.c.f. $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty(\mathtt{U}\times\mathtt{Y})$ of P with these particular $\tilde{X}$ and $\tilde{Y}$. Moreover, for any such d.c.f., we have

$$\begin{bmatrix} I & -R_{11} & -R_{12} \\ -P & I & 0 \\ 0 & -R_{21} & I-R_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Y\tilde{N}+I & M\tilde{Y} & * \\ X\tilde{N} & N\tilde{Y}+I & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} M\tilde{X} & Y\tilde{M} & * \\ N\tilde{X} & X\tilde{M} & * \\ * & * & * \end{bmatrix}, \quad (3.2)$$

where R is any stabilizing controller with internal loop that is equivalent to $\tilde{X}^{-1}\tilde{Y}$. If

$R = \left[\begin{smallmatrix} 0 & I \\ \tilde{Y} & I-\tilde{X} \end{smallmatrix}\right]$, *then*

$$\begin{bmatrix} I & -R_{11} & -R_{12} \\ -P & I & 0 \\ 0 & -R_{21} & I-R_{22} \end{bmatrix}^{-1} = \begin{bmatrix} M\tilde{X} & M\tilde{Y} & M \\ N\tilde{X} & N\tilde{Y}+I & N \\ M\tilde{X}-I & M\tilde{Y} & M \end{bmatrix} \tag{3.3}$$

(Thus, any such stabilizing R' is actually a canonical controller. However, also some non-canonical functions $R' = \left[\begin{smallmatrix} 0 & I \\ \tilde{Y} & I-\tilde{X} \end{smallmatrix}\right] \in \mathrm{H}^\infty$ do lead to (3.2) but the $*$'s (which denote unimportant entries) do not become stable unless $\tilde{X}$ and $\tilde{Y}$ are l.c.)

We record here an obvious consequence of the dual of Lemma 3.5:

COROLLARY 3.6. *Let P be a proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function. If a canonical controller YX^{-1} stabilizes P, then P has a r.c.f. and any r.c.f. NM^{-1} of P satisfies $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] \in \mathcal{G}\,\mathrm{H}^\infty$. Conversely, if there exists $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] \in \mathcal{G}\,\mathrm{H}^\infty$ such that $M(0) \in \mathcal{GB}(\mathtt{U})$ and $P = NM^{-1}$, then YX^{-1} stabilizes P.*

If P is strictly proper, then the X in Corollary 3.6 is necessarily invertible at 0:

LEMMA 3.7. *If P is strictly proper, then any stabilizing controller with internal loop for P is proper.*

(This follows from Lemmata 3.2, 3.3 and 3.5, because $P(0) = 0 \Rightarrow N(0) = 0 \Rightarrow \tilde{X}(0)M(0) = I \Rightarrow \tilde{X}(0) = M(0)^{-1} \in \mathcal{GB}(\mathtt{U})$.)

Example. The function $P = NM^{-1} = (1-z)^{-1}$, where $N = 1$, $M = 1-z$, has the stabilizing controller $R := \left[\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right]$ with internal loop (the canonical controller $\tilde{\mathscr{X}} = 0$, $\tilde{\mathscr{Y}} = -1$). Since $1-R_{22}$ is nowhere invertible, this controller is not equivalent to any proper controller, by Lemma 3.2. However, by Theorem 1.1, there are also proper stabilizing controllers for P (e.g., $Q(z) = -z$). The continuous-time equivalent of this example was presented in [32, p. 6], where the non-proper controller was shown to be the natural engineering solution (short circuit tracking) for the problem.

Notes for Section 3: The "if" part of Lemma 3.2 is from [32]. With the additional assumption that P has a d.c.f., Lemma 3.3 is contained in [4]. However, the proof in [4] is seven pages long, so we present a short, self-contained proof in Appendix A. Also most of Lemma 3.5 can be found in [4]. For further similar results, see [11]; for practical examples, see [32] and [4]. Lemma 3.7 becomes less obvious and even more important in the continuous-time setting of Theorem 7.3.

4. Partial feedback. In this section we treat *Dynamic Partial Feedback (DPF)*, where the controller Q sees only a part of the output and can affect only a part of the input.

Throughout this section we assume that P is a proper $\mathcal{B}(\mathtt{U}\times\mathtt{W},\mathtt{Z}\times\mathtt{Y})$-valued function. A proper $\mathcal{B}(\mathtt{Y},\mathtt{U})$-valued function Q is called a *stabilizing DPF-controller* for P if $\left[\begin{smallmatrix} 0 & Q \\ 0 & 0 \end{smallmatrix}\right]$ is a stabilizing controller for P. This obviously corresponds to Figure 1.2. Analogously, a $\mathcal{B}(\mathtt{Y}\times\Xi,\mathtt{U}\times\Xi)$-valued proper function R is called a *stabilizing DPF-controller with internal loop* for P if

$$R_{\mathrm{DPF}} := \begin{bmatrix} 0 & R_{11} & R_{12} \\ 0 & 0 & 0 \\ 0 & R_{21} & R_{22} \end{bmatrix} \qquad \text{(whose values lie in } (\mathtt{Z}\times\mathtt{Y}\times\Xi,\mathtt{U}\times\mathtt{W}\times\Xi)\text{)} \tag{4.1}$$

is a stabilizing controller with internal loop for P; see Figure 4.1. If such an R exists, then we call P *DPF-stabilizable* (with internal loop). (Further details are given in [11, Section 7.3]. Observe that the [DPF-]controller $R = \left[\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with internal loop functions exactly as the [DPF-]controller Q; we identify the two.)

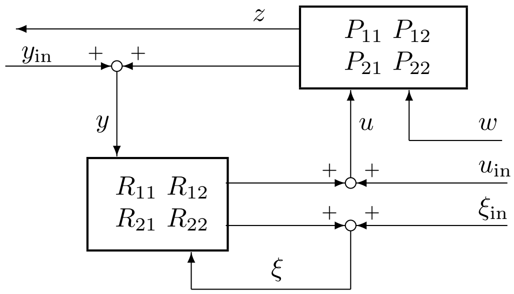


FIG. 4.1. *DPF-controller R with internal loop for P*

We call two stabilizing controllers with internal loop for P (say, R and R') *equivalent* if they lead to same maps $u_{\rm in}, y_{\rm in} \mapsto u, y$ (or equivalently, to same maps $u_{\rm in}, w, y_{\rm in} \mapsto u, y, z$, or equivalently, if $R_{\rm DF}$ and $R'_{\rm DF}$ are equivalent for P, or equivalently, if R and R' are equivalent for P_{21}; see [11, Lemma 7.3.8] for this equivalence).

DPF is the standard setting in the general (four-block) H^∞ regulator problem (see [11, Chapter 12] for this general case with internal loops).

Note that the second input (column) of P is the exogenous input (or disturbance) and the first input (column) is the one connected to the controller output, not vice versa (both variants can be found in the literature; the other choice would move the I's out of the diagonal in Theorem 4.2 below).

With the aid the Theorem 1.1, we can derive the following two theorems, which are direct generalizations of well-known results for rational functions. The first theorem reduces DPF-stabilization problems to ordinary dynamic stabilization problems:

THEOREM 4.1 (P iff P_{21}). *Assume that P is DPF-stabilizable. Then a proper $\mathcal{B}(\mathtt{Y} \times \Xi, \mathtt{U} \times \Xi)$-valued function R is a stabilizing DPF-controller with internal loop for P iff R is a stabilizing controller with internal loop for P_{21}.*

In particular, a proper $\mathcal{B}(\mathtt{Y}, \mathtt{U})$-valued function Q is a stabilizing DPF-controller for P iff Q is a stabilizing controller for P_{21}. It also follows that every stabilizing DPF-controller with internal loop for P is equivalent to one of the canonical controllers (for P_{21}) given by the Youla parameterization.

Observe that $P_{21} : u \mapsto y - y_{\rm in}$ is the control-to-measurement part of P.

A function is DPF-stabilizable iff it has a coprime factorization "through P_{21}":

THEOREM 4.2 (DPF). *The following are equivalent:*

(i) *P has a strictly proper stabilizing DPF-controller.*

(ii) *P has a proper stabilizing DPF-controller.*

(iii) *P has a stabilizing DPF-controller with internal loop.*

(iv) *P has a r.c.f. of the form $P = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ 0 & I \end{bmatrix}^{-1}$ such that N_{21} and M_{11} are r.c.*

(v) *P has a l.c.f. of the form $P = \begin{bmatrix} I & \tilde{M}_{12} \\ 0 & \tilde{M}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix}$ such that $\tilde{N}_{21}$ and $\tilde{M}_{22}$ are l.c.*

Assume (iv) and (v). Then $N_{21}M_{11}^{-1}$ is a r.c.f. of P_{21} and $\tilde{M}_{22}^{-1}\tilde{N}_{21}$ is a l.c.f. of P_{21}.

(Thus, the above r.c.f. of P contains a r.c.f. of P_{21} that can be used for the Youla parameterization of all stabilizing DPF-controllers with internal loop for P, by Theorem 4.1.)

One can also derive sufficient conditions for DPF-stabilizability in terms of realizations. One sufficient condition is the power-stabilizability and detectability of the subsystem corresponding to P_{21}; see [11, Lemma 7.3.6(c)] for details.

Notes for Section 4: For rational matrix-valued functions, the above two theorems can be found in, e.g., [6] or [8]; most of them were extended to the Callier–Desoer class in [5].

If (iv) holds, then the stabilizing DPF-controllers for P are, modulo equivalence, exactly the canonical controllers $\tilde{X}^{-1}\tilde{Y}$ for any $\tilde{X}, \tilde{Y} \in \mathrm{H}^\infty$ such that $\tilde{X}M_{11} - \tilde{Y}N_{21} = I$, by Lemma 3.5. An equivalent characterization is: those YX^{-1} for which $\begin{bmatrix} M_{11} & M_{12} & Y \\ M_{21} & M_{22} & 0 \\ N_{21} & N_{22} & X \end{bmatrix} \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U} \times \mathtt{W} \times \mathtt{Y})$ for some (hence any) r.c.f. NM^{-1} of P [11, Lemma 7.3.22]. A third equivalent characterization is the Youla parameterization (for P_{21}). By Theorem 4.1, also all proper stabilizing DPF-controllers for P are contained in any of these (as in Theorem 1.1).

The coprimeness condition in (iv) cannot be weakened: the r.c.f. $P = NM^{-1} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (z+1)/(z+2) & 0 \\ 0 & 1 \end{bmatrix}^{-1}$ is of the form $\begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}^{-1}$ (hence P and P_{21} both have a d.c.f. and thus have stabilizing controllers), but yet P is not DPF-stabilizable with internal loop, since $P_{11} = (z+2)/(z+1)$ is unstable ($\not\in \mathrm{H}^\infty$) and unaffected by any DPF-controller, because $P_{21} = 0$. However, if some P is DPF-stabilizable, then any r.c.f. of that P of the form $P = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ 0 & I \end{bmatrix}^{-1}$ has N_{21}, M_{11} r.c. [11, Corollary 7.3.17]. A corresponding continuous-time example is given by $P(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s/(s+1) & 0 \\ 0 & 1 \end{bmatrix}^{-1}$.

A rational right factorization is r.c. iff M has no other zeros than the poles of P. The coprimeness condition on N_{21} and M_{11} says that as we multiply the zeros of P away by M_{11}, we do not introduce to $N_{21} = P_{21}M_{11}$ any new zeros (in addition to those of P_{21}), i.e., that the poles of P are also poles of P_{21}. In other words, this says that the poles of P are visible through P_{21}; other kind of poles of P could not be stabilized by partial feedback having access to P_{21} only, as is the case in the above example $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (z+1)/(z+2) & 0 \\ 0 & 1 \end{bmatrix}^{-1}$.

Using the above two theorems and the other results in this article, [16] and [14], one could generalize to nonrational functions also the other classical results, as presented in, e.g., [6] or [8]. Part of this can be found in [11], whose Hypothesis 7.3.15 holds iff P is DPF-stabilizable with internal loop, by Theorem 4.2. This simplifies §7.3 of [11] significantly; similarly, Theorem 1.1 simplifies §7.1 and §7.2. Partially the same applies to state-space results.

5. Power stabilization. One sometimes wants to power-stabilize systems or transfer functions (or stabilize exponentially in the continuous-time setting). In this section we observe that the "power-variants" of our results hold and follow easily. (However, from the "power-variants" one cannot obtain the original results. Moreover, in the power-stabilization of systems, there are some results whose nonpower-stabilization variants are false.)

We write $N \in \mathrm{H}^\infty_{\text{power}}$ if $N(r\cdot) \in \mathrm{H}^\infty$ for some $r > 1$ (i.e., $N \in \mathrm{H}^\infty$ has a holomorphic extension to an open disc that contains $\overline{\mathbb{D}}$). We define *power*-variants of the following definitions by replacing H^∞ by $\mathrm{H}^\infty_{\text{power}}$: r.c., l.c., r.c.f., l.c.f., d.c.f., α-r.c.f., α-l.c.f., α-d.c.f., stabilizing [DPF-]controller, stabilizing [DPF-]controller with internal loop, canonical controller, DPF-stabilizable. Thus, e.g., a d.c.f. $\begin{bmatrix} M & Y \\ N & X \end{bmatrix}$ is a *power-d.c.f.* iff $\begin{bmatrix} M & Y \\ N & X \end{bmatrix}, \begin{bmatrix} M & Y \\ N & X \end{bmatrix}^{-1} \in \mathrm{H}^\infty_{\text{power}}$, and Q is a *power-stabilizing* controller for P iff $\begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} \in \mathrm{H}^\infty_{\text{power}}$.

Remark 5.1 (Power stabilization). *With the above "power-"concepts in place of the original ones and* $\mathrm{H}^\infty_{\text{power}}$ *in place of* H^∞*, Theorem 2.1, Lemmata A.5, A.10 and A.11 and the results of §1, §3 and §4 hold.*

(Also the power form of Proposition 2.2 holds in the sense explained in [16].)

Proof. This follows easily from the original results. E.g., if P has a power-stable r.c.f. NM^{-1}, then $P(r\cdot) = N(r\cdot)M(r\cdot)^{-1}$ is a r.c.f. for some $r > 1$, hence then $P(r\cdot)$ has a proper stabilizing controller $\tilde{Q}$, hence $Q := \tilde{Q}(r^{-1}\cdot)$ is power-stabilizing for P (because $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & -\tilde{Q} \\ -P(r\cdot) & I \end{smallmatrix}\right]^{-1}(r^{-1}\cdot) \in \mathrm{H}^\infty_{\mathrm{power}}$), by Theorem 1.1. $\square$

Observe that a rational function is in H^∞ iff it does not have a pole in $\overline{\mathbb{D}}$, or equivalently, iff it is in $\mathrm{H}^\infty_{\mathrm{power}}$. Similarly, also finite-dimensional state-space stability coincides with state-space power stability. In the infinite-dimensional setting, both forms of stability are very popular.

6. Non-proper controller functions. In this section we study "stabilizing controllers" of the form of a possibly non-proper function. We also show that all such controllers are canonical controllers and we explain how they relate to each other.

In the matrix-valued case, a factorization NM^{-1} is well defined everywhere on $\mathbb{D}$ except possibly for some isolated points (assuming that $M, N \in \mathrm{H}^\infty$, $\det M \not\equiv 0$). In the operator-valued case, one may easily end up with functions having disconnected domains. Moreover, in dynamic stabilization one often meets the question whether two functions can be identified when they coincide on the intersection of their domains.

We show that if a function Q stabilizes P in a reasonable sense, then $Q = YX^{-1}$, where YX^{-1} is a stabilizing canonical controller for P in the standard sense, and then YX^{-1} (on the subset of $\mathbb{D}$ where X^{-1} exists) is the maximal holomorphic extension of Q (within $\mathbb{D}$). A similar claim holds for P. It follows that any such function element of Q stabilizes any such function element of P (on the intersection of their domains) in the same sense.

LEMMA 6.1. *Let $\Omega \subset \mathbb{D}$ be open and let $\alpha, \beta \in \Omega$. Let $P : \Omega \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ and $Q : \Omega \to \mathcal{B}(\mathtt{Y}, \mathtt{U})$.*

Then some $E \in \mathrm{H}^\infty(\mathtt{Y} \times \mathtt{U})$ satisfies $E\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right] = I = \left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]E$ on Ω iff there exists $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$ such that $M(z), X(z) \in \mathcal{GB}$ for all $z \in \Omega$ and $P = NM^{-1}$ and $Q = YX^{-1}$ on Ω.

If such a quadruple $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$ exists, and $P = N_1M_1^{-1}$ and $Q = Y_1X_1^{-1}$ are α-r.c.f.'s, then they are β-r.c.f.'s and $\left[\begin{smallmatrix} M_1 & Y_1 \\ N_1 & X_1 \end{smallmatrix}\right] \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$. Moreover, then any holomorphic extension (to a connected open subset of $\mathbb{D}$) of any restriction of NM^{-1} (to an open set) is a restriction of NM^{-1} (with domain $\{z \in \mathbb{D} \mid M(z)^{-1}$ exists$\}$).

(Obviously, then $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right]$ is a β-d.c.f. of Q.)

We observed above that even if the domain of P and Q is not connected, a single d.c.f. applies at each component of the domain (if Q stabilizes P "at each component with the same inverse E"; otherwise the different components of P could be arbitrary). Moreover, there is no problem of extending P or Q holomorphically within the unit disc (the values of the functions at a certain point do not depend of the domain). This is an alternative proof of the fact that the function $P = \log$ (or any other function with different branches) is not dynamically stabilizable.

Next we define (possibly non-proper) stabilizing controller functions. Assume, for a while, that $\Omega_P \subset \mathbb{D}$ is open and connected. Let $P : \Omega_P \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ be holomorphic. If $\Omega \subset \Omega_P$ is open and $Q : \Omega \to \mathcal{B}(\mathtt{Y}, \mathtt{U})$ is holomorphic, then we call Q a *stabilizing controller function for P* if $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$.[4] We define *stabilizing DPF-controller*

[4]Naturally, this means that there exists $E \in \mathrm{H}^\infty$ such that $E\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right] \equiv I \equiv \left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]E$ on Ω. Note that if(f) $0 \in \Omega$, then this is equivalent to the definition of a proper stabilizing controller. Thus, "proper stabilizing controller function" means the same as "proper stabilizing controller".

functions analogously (i.e., we do not require them to be proper).

By Lemma 6.1, the above definitions are in a complete accordance with the old ones and any stabilizing controller function is a stabilizing canonical controller. In particular, the existence of a stabilizing controller function (for a proper function P) is equivalent to Theorem 1.1(i). However, not all canonical controllers are functions, as one observes from the example at the end of §3.

7. Continuous-time results. In this section we shall show that almost all *DT* (discrete-time) results of the other sections also hold in their *CT* (continuous-time) forms, even if we use the standard continuous-time properness (defined later below). But first we record the following obvious consequence of the Riemann Mapping Theorem:

REMARK 7.1 (Cayley). *The results in Sections 1–4, 6 and A except Proposition 2.2 hold true even if we replace $\mathbb{D}$ by any simply connected open $\mathbb{D}' \subsetneq \mathbb{C}$ and the origin 0 by any $\zeta \in \mathbb{D}'$.*

(We shall often use this implicitly when referring to those results. In fact, many of these results would hold even if $\mathbb{D}'$ was not simply connected (some results would hold with essentially the same proof, some others could be reduced to the simply connected case if, e.g., $\mathbb{D}'$ is a finite union of simply connected open sets containing ζ).)

In the most important special case, where $\mathbb{D}' := \mathbb{C}^+$ and $\zeta \in \mathbb{C}^+$, we can use the "Cayley" mapping $f : s \mapsto \frac{\zeta - s}{\zeta + s}$ to map $\mathbb{C}^+ \to \mathbb{D}$ conformally with $\zeta \mapsto 0$. Then we can apply the earlier results to $P \circ f^{-1}$, $M \circ f^{-1}$, $N \circ f^{-1}$ etc. in place of P, M, N etc.

In CT, the right half-plane $\mathbb{C}^+$ takes the role of $\mathbb{D}$. Therefore, for the rest this section, we redefine some concepts (cf. Theorem 7.3):

DEFINITION 7.2 (CT forms). *Given $\omega \in \mathbb{R}$ we set $\mathbb{C}^+_\omega := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega\}$, and by $\mathrm{H}^\infty_\omega(\mathtt{U}, \mathtt{Y})$ we denote the Banach space of bounded holomorphic functions $\mathbb{C}^+_\omega \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ with the supremum norm.*

We call P proper *if $P \in \mathrm{H}^\infty_\infty := \cup_{\omega \in \mathbb{R}} \mathrm{H}^\infty_\omega$, i.e., if P is a bounded holomorphic function on some right half-plane. (We identify a holomorphic function on a right half-plane $\mathbb{C}^+_\omega$ with its restriction to any open subset of $\mathbb{C}^+_\omega$.) It is* strictly proper *if, in addition, $P(z) \to 0$ as $\operatorname{Re} z \to +\infty$.*[5] *We set $\mathbb{C}^+ := \mathbb{C}^+_0$, $\mathrm{H}^\infty := \mathrm{H}^\infty_0$. Moreover, in all definitions and results in the other sections, we replace $\mathbb{D}$ by $\mathbb{C}^+$ and invertibility at 0 by the existence of a proper inverse.*

(The main motivation for the above properness concept is that a function is proper iff it is the transfer function of a well-posed linear system [21].)

Thus, e.g., if $N, M \in \mathrm{H}^\infty$ are r.c., $M^{-1} \in \mathrm{H}^\infty_\infty(\mathtt{U})$ and $P = NM^{-1}$ (on a right half-plane), then we call $P = NM^{-1}$ an *r.c.f. of P*; similarly, if P and Q are proper and $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$, then Q is a *proper stabilizing controller* for P. Recall that $(I - P^R)^{-1} \in \mathrm{H}^\infty$ in (3.1) means that $(I - P^R)^{-1}$ is the restriction of some element of H^∞, or equivalently, that some $E \in \mathrm{H}^\infty$ satisfies $E(I - P^R) = I = (I - P^R)E$ on some right half-plane (since P and R were assumed to be proper in (3.1)). If $I - R_{22}$ has a proper inverse, then we again (see below Lemma 3.2) identify R with the proper controller $R_{11} + R_{12}(I - R_{22})^{-1}R_{21}$.

When N and M are r.c., $\alpha \in \mathbb{C}^+$, $M(\alpha) \in \mathcal{GB}(\mathtt{U})$ and $P = NM^{-1}$ on a neighborhood of α, we call $P = NM^{-1}$ an *α-r.c.f. of P*.

[5] This means that for each $\epsilon > 0$ there exists $\omega_\epsilon \in \mathbb{R}$ such that $\|P(z)\| < \epsilon$ for all $z \in \mathbb{C}$ such that $\operatorname{Re} z > \omega_\epsilon$.

Next we define (possibly non-proper) stabilizing controller functions. Let, for a while, $\Omega_P \subset \mathbb{C}^+$ be open and connected. Let $P : \Omega_P \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ be holomorphic. If $\Omega \subset \Omega_P$ is open and $Q : \Omega \to \mathcal{B}(\mathtt{Y}, \mathtt{U})$ is holomorphic, then we call Q a *stabilizing controller function for* P if $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$.[6] We define *stabilizing DPF-controller functions* analogously (i.e., we do not require them to be proper, whereas we still require that a *stabilizing [DPF-]controller with internal loop* is determined by a proper function R, as above).

By the arguments of Remark 7.1, the corresponding DT comments (below Lemma 6.1) apply here too (with Theorem 7.4 in place of Theorem 1.1).

THEOREM 7.3 (CT forms). *Propositions A.1 and A.2, Lemmata A.3, A.4, A.9, A.10 and A.11 and the results in Sections 1–3 and 6 hold in their CT forms too if we replace Theorem 1.1 (resp., 2.1, Lemma 3.2) by Theorem 7.4 (resp., 7.5, Lemma 7.7).*

See Theorem 7.8 (resp., 7.9, Remark 7.10) for Theorem 4.1 (resp., Theorem 4.2, Remark 5.1). See [16] for the CT definitions for Proposition 2.2.

The main Theorem 1.1 holds in its CT form too once we remove "[strictly] proper" from (i) and (ii). We write this explicitly below with a new condition (i).

THEOREM 7.4 (CT: Dynamic feedback stabilization). *Let P be a proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function, i.e., $P \in \mathrm{H}^\infty_\omega(\mathtt{U}, \mathtt{Y})$ for some $\omega \ge 0$. Let $\zeta \in \mathbb{C}^+_\omega$. Then following are equivalent:*

(i) *There exists a holomorphic $\mathcal{B}(\mathtt{Y}, \mathtt{U})$-valued function Q on a neighborhood of ζ such that $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$.*

(ii) *P has a stabilizing controller function.*

(iii) *P has a stabilizing controller with internal loop.*

(iv) *P has a r.c.f.*

(v) *$\left[\begin{smallmatrix} P & 0 \\ 0 & I_{\mathtt{Z}} \end{smallmatrix}\right]$ has a r.c.f. for some (hence any) Hilbert space $\mathtt{Z}$.*

Assume that P has a r.c.f. $P = NM^{-1}$. Then $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{U} \times \mathtt{Y})$ can be extended to an invertible element of $\mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$, say $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$. Denote its inverse by $\left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right] \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$. Then all stabilizing controllers for P are given by the Youla(–Bongiorno) parameterization

$$Q = (Y + MV)(X + NV)^{-1} \qquad (= (\tilde{X} + V\tilde{N})^{-1}(\tilde{Y} + V\tilde{M})), \tag{7.1}$$

where $V \in \mathrm{H}^\infty(\mathtt{Y}, \mathtt{U})$ is arbitrary (the controller is proper iff $(X + NV)^{-1}$ is proper, or equivalently, iff $(\tilde{X} + V\tilde{N})^{-1}$ is proper). The map $V \mapsto Q$ is one-to-one.

If P is strictly proper, then all these controllers are proper.

(Note that in PDE systems, the transfer function is usually strictly proper. That is also the case for well-posed systems having a bounded input or output operator and no feedthrough [13, Theorem 1.2].)

If, in Theorem 7.4, we set $\Omega_P := \mathbb{C}^+_\omega$ and fix some $\alpha, \beta \in \Omega_P$, then also the six conditions listed below become equivalent to (i):

THEOREM 7.5. *Assume that $\Omega_P \subset \mathbb{C}^+$ is open and connected and contains a right half-plane, $P : \Omega_P \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ is holomorphic and proper, and $\alpha, \beta \in \Omega_P$.*

Then the conditions (ii'), (ii"), (iii'), (iv'), (iv") and (iv"') of Theorem 2.1 are equivalent to (iv) of Theorem 7.4. Moreover, then any α-r.c.f. of P is a β-r.c.f. of P. The same holds with "l.c.f." or "d.c.f." in place of "r.c.f.".

[6] Naturally, this means that there exists $E \in \mathrm{H}^\infty$ such that $E\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right] \equiv I \equiv \left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right] E$ on Ω. Note that if(f) Ω contains a right half-plane and P and Q are proper, then this is equivalent to the definition of a proper stabilizing controller.

(See Proposition 2.2 and Corollary 1.2 for further equivalent conditions.)

Condition (ii") says that for any point α in the domain Ω_P of P, there exists a stabilizing controller function whose domain includes α. We do not know whether a proper stabilizing controller always exists even if we assume that P is proper. Naturally, a similar comment applies to Theorem 7.9. In the matrix-valued case, a proper stabilizing controller $Q \in \mathrm{H}^\infty$ exists, by Corollary 1.2(vi) (through Theorem 7.3). If P is strictly proper, then any stabilizing controller (with or without internal loop) for P is proper, by Lemma 3.7. Moreover, whenever P has a sufficiently regular right factorization, a strictly proper Q exists:

Theorem 7.6 (CT: strictly proper Q). *Assume that P has a r.c.f. and that $P = NM^{-1}$, where $N \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{Y})$, $M \in \mathrm{H}^\infty(\mathtt{U})$ and M^{-1} is proper. If $M(+\infty) := \lim_{\operatorname{Re} s \to +\infty} M(s)$ exists, then there exists a strictly proper Q such that $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty(\mathtt{Y} \times \mathtt{U})$.*

(Note that this NM^{-1} need not be a r.c.f.; the existence of a r.c.f. is only needed for guaranteeing the existence of a stabilizing controller.)

A r.c.f. of P (if any exists) can be determined from the LQR Riccati equation for an output-stabilizable realization, as the resulting closed-loop transfer function $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$; see [2] or [14]. For sufficient regularity (for Theorem 7.6) of this particular factorization, many different assumptions can be found in the literature, such as the analytic semigroup setting of [10] or certain assumptions on the unboundedness of the control and/or observation operators [30] [11].

Next we rewrite Lemma 3.2, which says that a controller with internal loop is proper iff $(I - R_{22})^{-1}$ is proper:

Lemma 7.7 (CT: proper R). *Assume that $R = \left[\begin{smallmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{smallmatrix}\right]$ is a stabilizing controller with internal loop for P.*

Then R is equivalent to a stabilizing controller with internal loop of form $\tilde{R} = \left[\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right]$ iff $(I - R_{22})^{-1} \in \mathrm{H}^\infty_\infty$. If $(I - R_{22})^{-1} \in \mathrm{H}^\infty_\infty$, then the unique solution is given by $Q = R_{11} + R_{12}(I - R_{22})^{-1} R_{21}$.

(This holds by the original proof (with H^∞_∞-invertibility in place of invertibility at the origin).)

As in Theorem 4.1, we can reduce DPF-stabilization problems to ordinary dynamic stabilization problems:

Theorem 7.8 (CT: P iff P_{21}). *Assume that P is a proper $\mathcal{B}(\mathtt{U} \times \mathtt{W}, \mathtt{Z} \times \mathtt{Y})$-valued DPF-stabilizable function. Then a proper $\mathcal{B}(\mathtt{Y} \times \Xi, \mathtt{U} \times \Xi)$-valued function R is a stabilizing DPF-controller with internal loop for P iff R is a stabilizing controller function with internal loop for P_{21}.*

Moreover, a $\mathcal{B}(\mathtt{Y}, \mathtt{U})$-valued function Q is a stabilizing DPF-controller function for P iff Q is a stabilizing controller fucntion for P_{21}. It also follows that every stabilizing DPF-controller with internal loop for P is equivalent to one of the canonical controllers (for P_{21}) given by the Youla parameterization.

(Also Theorem 4.1 holds in this CT terminology (and vice versa); the only difference is that here we do not require Q to be proper.)

As in Theorem 4.2, a function P is DPF-stabilizable iff it has a coprime factorization "through P_{21}":

Theorem 7.9 (CT: DPF). *The following are equivalent for a proper $\mathcal{B}(\mathtt{U} \times \mathtt{W}, \mathtt{Z} \times \mathtt{Y})$-valued function P:*

(ii) *P has a stabilizing DPF-controller function.*
(iii) *P has a stabilizing DPF-controller with internal loop.*

(iv) *P has a r.c.f. of the form $P = \left[\begin{smallmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} M_{11} & M_{12} \\ 0 & I \end{smallmatrix}\right]^{-1}$ such that N_{21} and M_{11} are r.c.*

(v) *P has a l.c.f. of the form $P = \left[\begin{smallmatrix} I & \tilde M_{12} \\ 0 & \tilde M_{22} \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} \tilde N_{11} & \tilde N_{12} \\ \tilde N_{21} & \tilde N_{22} \end{smallmatrix}\right]$ such that $\tilde N_{21}$ and $\tilde M_{22}$ are l.c.*

Assume (iv) and (v). Then $N_{21}M_{11}^{-1}$ is a r.c.f. of P_{21} and $\tilde M_{22}^{-1}\tilde N_{21}$ is a l.c.f. of P_{21}.

(The changes above are essentially the same as those in Theorem 7.4. Naturally, by Theorem 7.8, we could add a condition resembling Theorem 7.4(i).)

Also Remark 5.1 holds in CT, mutatis mutandis:

Remark 7.10 (Exponential stabilization). *Define the power concepts above Remark 5.1 with $\mathrm{H}^\infty_{\mathrm{exp}} := \cup_{\omega<0}\mathrm{H}^\infty_\omega$ in place of $\mathrm{H}^\infty_{\mathrm{power}}$.*

With such "power-"concepts in place of the original ones and $\mathrm{H}^\infty_{\mathrm{exp}}$ in place of H^∞, Theorems 7.4–7.9 and the CT forms (see Definition 7.2, Theorem 7.3 and the text between them) of Corollary 1.2, Theorem 2.1, Lemmata A.10 and A.11 and the results of Section 3 hold if we rewrite the CT form of Corollary 1.2(vii) as follows:

(vii) *$P = NM^{-1}$, where $N, M \in \mathrm{H}^\infty_{\mathrm{exp}}$, $N^*N + M^*M \ge \epsilon I$ on $\mathbb{C}^+_{-\epsilon}$, $\epsilon > 0$ and* $\det M \not\equiv 0$.

(Also the "power form" of Proposition 2.2 holds in the sense explained in [15]. Thus, all results in Sections 1–4 are covered with some slight modifications.)

In the CT terminology for the "power concepts" of Section 5, one usually replaces the component "power-" by the word "exponential[ly]" (see, e.g., [24] or [11] for details).

Despite the "different properness and different power stability" in CT, the "same" results hold as in DT, with the exception that we do not guarantee the existence of a proper stabilizing [DPF-]controller in general (just in the three special cases mentioned below Theorem 7.5) and we made the slight "change" ($\mathbb{C}^+_{-\epsilon}$ instead of $\mathbb{C}^+$) in (vii) at the end of Remark 7.10.

Appendix A. Discrete-time proofs.

In this appendix we shall prove all our nontrivial results except those of Section 7. We start by showing that every dynamically stabilizable function has a r.c.f. For that purpose we need to recall part of [16], particularly the fact that any $\mathrm{H}^\infty / \mathrm{H}^\infty$ fraction can be written as a fraction of so called "weakly r.c. functions". This requires the following definitions.

If $N \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{Y})$, $M \in \mathrm{H}^\infty(\mathtt{U})$ and $M(0) \in \mathcal{GB}(\mathtt{U})$, then we call NM^{-1} a *right factorization* (*of P*, if $P = NM^{-1}$ near 0). We call such a factorization a *weakly right coprime factorization (w.r.c.f.)* if, in addition,

$$\begin{bmatrix} N \\ M \end{bmatrix} f \in \mathrm{H}^2 \;\Rightarrow\; f \in \mathrm{H}^2 \tag{A.1}$$

for every proper $\mathtt{U}$-valued function f; i.e., if a holomorphic $\mathtt{U}$-valued function f defined on a neighborhood of 0 is a restriction of an element of $\mathrm{H}^2(\mathtt{U})$ whenever $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f$ is a restriction of an element of $\mathrm{H}^2(\mathtt{Y} \times \mathtt{U})$.

We recall the following two propositions from [16]:

Proposition A.1 (W.r.c.f.). *A $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function P has a right factorization iff it has a weakly right coprime factorization.*

Moreover, if $P = NM^{-1}$ is a w.r.c.f., then all right factorizations of P are parameterized by $P = (NV)(MV)^{-1}$, where $V \in \mathrm{H}^\infty(\mathtt{U})$ and V^{-1} is proper. The *w.r.c.f.'s are those for which* $V^{-1} \in \mathrm{H}^\infty$ *too. In particular, if a function* P *has an r.c.f., then every w.r.c.f. of* P *is a r.c.f.*

PROPOSITION A.2. *If* NM^{-1} *is a w.r.c.f. with* $M \in \mathrm{H}^\infty(\mathtt{U})$, R *is a proper* $\mathcal{B}(\mathtt{Z},\mathtt{U})$*-valued function, and* $NR, MR \in \mathrm{H}^\infty$, *then* $R \in \mathrm{H}^\infty$.

LEMMA A.3. *If* $P = NM^{-1}$ *is a w.r.c.f. and* $(I-P)^{-1} \in \mathrm{H}^\infty$, *then* $(M-N)^{-1} \in \mathrm{H}^\infty$.

Proof. Since $I - P = (M-N)M^{-1}$, we have $M(M-N)^{-1} = (I-P)^{-1} \in \mathrm{H}^\infty$ and

$$N(M-N)^{-1} = NM^{-1}M(M-N)^{-1} = P(I-P)^{-1} = (I-P)^{-1} - I \in \mathrm{H}^\infty . \quad \text{(A.2)}$$

By Proposition A.2, it follows that $(M-N)^{-1} \in \mathrm{H}^\infty$. ∎

Now we are ready to prove the implication (ii)⇒(iv) in Theorem 1.1.

LEMMA A.4. *If* Q *is a proper stabilizing controller to a proper* $\mathcal{B}(\mathtt{U},\mathtt{Y})$*-valued function* P, *then* P *has a r.c.f.*

In the proof we take the factorizations (of P and Q) determined by (1.1), replace them by weakly coprime factorizations ($P = NM^{-1}$ and $Q = YX^{-1}$) and use Lemma A.3 to prove the invertibility of $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$ in H^∞.

Proof of Lemma A.4: By (1.1), $P = N_1 M_1^{-1}$, where $M_1 := (I - QP)^{-1} \in \mathrm{H}^\infty$ and $N_1 := PM_1 \in \mathrm{H}^\infty$. By Proposition A.1, P has a w.r.c.f. $P = NM^{-1}$. Similarly, Q has a w.r.c.f. $Q = YX^{-1}$. It obviously follows that

$$\begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix} = \begin{bmatrix} 0 & Y \\ N & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & X \end{bmatrix}^{-1} \quad \text{(A.3)}$$

is a w.r.c.f. Since $\big(I - \left[\begin{smallmatrix} 0 & Q \\ P & 0 \end{smallmatrix}\right]\big)^{-1} \in \mathrm{H}^\infty$ (see (1.1)), it follows from Lemma A.3 that

$$\left(\begin{bmatrix} M & 0 \\ 0 & X \end{bmatrix} - \begin{bmatrix} 0 & Y \\ N & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} M & -Y \\ -N & X \end{bmatrix}^{-1} \in \mathrm{H}^\infty . \quad \text{(A.4)}$$

Therefore, $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]^{-1} = \big(\left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \left[\begin{smallmatrix} M & -Y \\ -N & X \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]\big)^{-1} \in \mathrm{H}^\infty$. ∎

(The main part of the above is essentially the proof of [11, Lemmata 7.1.5 & 6.6.6] with w.r.c.f.'s in place of r.c.f.'s.)

Next we observe that an r.c.f. can be extended to a d.c.f. that has $Y(0) = 0$ and $X(0) \in \mathcal{GB}$:

LEMMA A.5. *Let* NM^{-1} *be a* $\mathcal{B}(\mathtt{U},\mathtt{Y})$*-valued r.c.f. Then* NM^{-1} *has an d.c.f.* $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$ *such that* $X(0), \tilde{X}(0), \tilde{M}(0) \in \mathcal{GB}$ *and* $Y(0) = 0$. *Moreover, the strictly proper function* $Q := YX^{-1} = \tilde{X}^{-1}\tilde{Y}$ *stabilizes* $P := NM^{-1}$ *in the sense that* $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$. *Furthermore, we can have above* $X(0) = I = \tilde{M}(0)$ *(and* $\tilde{X}(0) = I = M(0)$ *if we replace* N *by* NV *and* M *by* MV, *where* $V := M(0) \in \mathcal{GB}(\mathtt{U})$*).*

Proof. By [16], we can extend the function $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ to a d.c.f. $\left[\begin{smallmatrix} M & Y_0 \\ N & X_0 \end{smallmatrix}\right]$. Set $\left[\begin{smallmatrix} M & Y_1 \\ N & X_1 \end{smallmatrix}\right] := \left[\begin{smallmatrix} M & Y_0 \\ N & X_0 \end{smallmatrix}\right] \left[\begin{smallmatrix} I & -M(0)^{-1}Y_0 \\ 0 & I \end{smallmatrix}\right]$ to have $Y_1(0) = 0$. It follows that $X_1(0)$ must be invertible. Set $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] := \left[\begin{smallmatrix} M & Y_1 \\ N & X_1 \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ 0 & X_1(0)^{-1} \end{smallmatrix}\right]$ to have $X(0) = I$, hence $\left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} M(0)^{-1} & 0 \\ P(0) & I \end{smallmatrix}\right]$, where $P := NM^{-1}$ (and $\left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right] := \left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]^{-1}$). The final claim obviously follows. One easily verifies that $\left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} M\tilde{X} & Y\tilde{M} \\ N\tilde{X} & X\tilde{M} \end{smallmatrix}\right] = I$. ∎

LEMMA A.6. *Conditions (i)–(v) of Theorem 1.1 are equivalent.*

Proof. 1° *(i), (ii), (iv) and (v):* By Lemmata A.4 and A.5, (i), (ii) and (iv) are equivalent. If $P = NM^{-1}$ is a r.c.f., then so is obviously $\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} N & 0 \\ 0 & I \end{smallmatrix}\right]\left[\begin{smallmatrix} M & 0 \\ 0 & I \end{smallmatrix}\right]^{-1}$; thus, (iv) implies (v). Assume then (v). By Proposition 2.2, $\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$ has a realization $\tilde{\Sigma}$ that satisfies the Finite Cost Condition and its dual. Remove the bottom row of $\tilde{\Sigma}$ to obtain a realization Σ of P that satisfies the Finite Cost Condition and its dual (this follows very easily). By Proposition 2.2, this shows that P has a r.c.f.

2° *(iii):* Trivially, (i) implies (iii) (take $R := \left[\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right]$). If (iii) holds, then so does (v), by Lemmata 3.1 and A.4 (with $\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$ in place of P). □

The following is well known [24]:

PROPOSITION A.7 (Schur decomposition). **(a)** *If* $A = B^{-1} \in \mathcal{GB}(\mathtt{X}_1 \times \mathtt{X}_2, \mathtt{Y}_1 \times \mathtt{Y}_2)$, *then* $A_{22} \in \mathcal{GB}(\mathtt{X}_2, \mathtt{Y}_2) \Leftrightarrow B_{11} \in \mathcal{GB}(\mathtt{Y}_1, \mathtt{X}_1) \;\Rightarrow\; B_{11}^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

(b) *If* $A \in \mathcal{B}(\mathtt{X}_1 \times \mathtt{X}_2, \mathtt{Y}_1 \times \mathtt{Y}_2)$ *and* $A_{22} \in \mathcal{GB}(\mathtt{X}_2, \mathtt{Y}_2)$, *then* $A \in \mathcal{GB} \Leftrightarrow A_{11} - A_{12}A_{22}^{-1}A_{21} \in \mathcal{GB}(\mathtt{X}_1, \mathtt{Y}_1)$.

Now we can prove the results of §3.

Proof of Lemma 3.2: 1° Set

$$A := I - P^R = \left[\begin{array}{cc|c} I & -R_{11} & -R_{12} \\ -P & I & 0 \\ \hline 0 & -R_{21} & I - R_{22} \end{array}\right], \tag{A.5}$$

$B := A^{-1}$. Assume that $A_{22}(0) := I - R_{22}(0) \in \mathcal{GB}$. By Proposition A.7(a), $B_{11}^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21} = \left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]$, where $Q := R_{11} - R_{12}(I - R_{22})^{-1}R_{21}$, hence $B_{11} = \tilde{B}_{11}$, where $\tilde{B} := (I - P^{\tilde{R}})^{-1}$, $\tilde{R} := \left[\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Thus, R and $\tilde{R}$ are "equivalent". But $\tilde{B}_{21} = 0, \tilde{B}_{12} = 0$ and $\tilde{B}_{22} = I$, hence $\tilde{B} \in \mathrm{H}^\infty$, so also $\tilde{R}$ is stabilizing.

2° Assume that R and $\tilde{R}$ are equivalent. Then we again have $B_{11} = \tilde{B}_{11}$. But $\tilde{B}_{11}^{-1} = \left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]$ is proper, hence $B_{11}(0) \in \mathcal{GB}$. By Proposition A.7(a), this implies that $\mathcal{GB} \ni A_{22}(0) = I - R_{22}(0)$. □

Proof of Lemma 3.3: By Lemma A.6, P has a r.c.f. NM^{-1}. Set

$$T := \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} := (I - P^R)^{-1}, \qquad \tilde{X} := M^{-1}T_{11}, \qquad \tilde{Y} := M^{-1}T_{12}. \tag{A.6}$$

Since $T(I-P^R) = I = (I-P^R)T$, we have (see (A.5)) $T_{11}I - T_{12}P = I$, $-PT_{11} + T_{21} = 0$ and $-PT_{12} + T_{22} = I$, hence $\tilde{X} - \tilde{Y}P = M^{-1}$, i.e., $\tilde{X}M - \tilde{Y}N = I$, and also $-N\tilde{X} + T_{21} = 0$, $-N\tilde{Y} + T_{22} = 0$. Thus, $N\tilde{X}, N\tilde{Y} \in \mathrm{H}^\infty$. But $M\tilde{X} = T_{11} \in \mathrm{H}^\infty$ and $M\tilde{Y} = T_{12} \in \mathrm{H}^\infty$, hence $\tilde{X}, \tilde{Y} \in \mathrm{H}^\infty$, by Proposition A.2, hence $\tilde{X}$ and $\tilde{Y}$ are l.c. By duality (because R^{d} is stabilizing for P^{d}), we get X and Y. □

The following is a direct consequence of Proposition A.7:

LEMMA A.8. *Let P be $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued and proper and let $R = \left[\begin{smallmatrix} 0 & I \\ \tilde{Y} & I - \tilde{X} \end{smallmatrix}\right]$ be $\mathcal{B}(\mathtt{Y} \times \mathtt{U}, \mathtt{Y} \times \mathtt{Y})$-valued and proper. Then $R(0) \in \mathcal{GB}$ iff $(\tilde{X} - \tilde{Y}P)(0) \in \mathcal{GB}$. Moreover, if $R(0) \in \mathcal{GB}$, then (set $M := (\tilde{X} - \tilde{Y}P)^{-1}$)*

$$(I - P^R)^{-1} = \begin{bmatrix} I + M\tilde{Y}P & M\tilde{Y} & M \\ P(I + M\tilde{Y}P) & I + PM\tilde{Y} & PM \\ M\tilde{Y}P & M\tilde{Y} & M \end{bmatrix} = \begin{bmatrix} M\tilde{X} & M\tilde{Y} & M \\ PM\tilde{X} & I + PM\tilde{Y} & PM \\ M\tilde{X} - I & M\tilde{Y} & M \end{bmatrix} \tag{A.7}$$

Proof of Lemma 3.5: 1° One easily verifies (3.3). By the definition of equivalence (and the equation $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]\left[\begin{smallmatrix} \tilde X & -\tilde Y \\ -\tilde N & \tilde M \end{smallmatrix}\right] = I$), (3.2) follows.

2° If $P = NM^{-1}$ is a r.c.f., $\tilde X, \tilde Y \in \mathrm{H}^\infty$ and $\Delta := \tilde X M - \tilde Y N \in \mathcal{G}\,\mathrm{H}^\infty$ (by Proposition A.1, the same then holds for all r.c.f.'s of P), then $N_1 := N\Delta^{-1}$, $M_1 := M\Delta^{-1}$ form a r.c.f. of P and satisfy $\tilde X M_1 - \tilde Y N_1 = I$. By Lemma 3.4 and (3.3), only the "only if" from the first "iff" remains to be proved.

3° Assume that R' is stabilizing. By the proof of Lemma 3.3, R' is equivalent to $\tilde X_1^{-1}\tilde Y_1$, where $\tilde X_1, \tilde Y_1 \in \mathrm{H}^\infty$ and $\tilde X_1 M_1 - \tilde Y_1 N_1 = I$ for some r.c.f. $P = N_1 M_1^{-1}$. By (A.7) and the equivalence of R' to $\tilde X_1^{-1}\tilde Y_1$, we have $\begin{bmatrix} M_1\tilde X_1 & M_1 \tilde Y_1\end{bmatrix} = \begin{bmatrix} M\tilde X & M\tilde Y\end{bmatrix}$, hence $\tilde X = \Delta \tilde X_1$ and $\tilde Y = \Delta \tilde Y_1$, where

$$\Delta := M^{-1}M_1 = \tilde X M_1 - \tilde Y N_1 \in \mathrm{H}^\infty(\mathtt{U}) \tag{A.8}$$

(because $M^{-1} = \tilde X - \tilde Y P$). But $\Delta^{-1} = M_1^{-1}M$, hence $M_1\Delta^{-1} = M$ and $N_1\Delta^{-1} = PM$. Since $M, PM \in \mathrm{H}^\infty$, by (A.7), we have $\Delta^{-1} \in \mathrm{H}^\infty$, by Proposition A.2. Thus, $\tilde X M_1 - \tilde Y N_1 = \Delta \in \mathcal{G}\,\mathrm{H}^\infty$. □

Now we are ready to complete the proof of our main result:

Proof of Theorem 1.1: Lemma A.6 contains the equivalence. Assume then that $P = NM^{-1}$ is a r.c.f. By [16], $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ can be extended to a d.c.f. It obviously follows that $\tilde M(0) \in \mathcal{GB}$ and $P = \tilde M^{-1}\tilde N$ (see [11] or [24]). The Youla parameterization is essentially from [4, Theorem 5.5] or [11, Theorem 7.2.14] (and can easily be obtained from Lemmata 3.3 and 3.5). The claim on properness is from Lemma 3.2 and that on strict properness is from Lemma 3.7. □

Proof of Theorem 2.1: 1° If $P = NM^{-1}$ is an α-r.c.f. and $\tilde X M - \tilde Y N = I$, $\tilde X, \tilde Y \in \mathrm{H}^\infty$, then $\tilde X - \tilde Y P = M^{-1}$ near α, hence on Ω_P, hence then $P = NM^{-1}$ is a β-r.c.f. too. The rest of the last paragraph follows analogously.

2° *(iii)–(iv"'):* By 1°, (iv) and (iv') are equivalent. The implication (iv')⇒(iv"') is from Theorem 1.1 (and Remark 7.1) and the converse is trivial. By duality, we get (iv")⇔(iv"'). The equivalence of (iii) and (iii') follows from Lemma 3.3.

3° As in Remark 7.1, we observe that Theorem 1.1 holds with α in place of 0, so the equivalence of (ii") and (iv') follows from that of (i) and (iv). Similarly, if (ii') holds and $z \in \Omega_Q$, then P has a z-r.c.f., hence then (iv) holds, by 1°. Trivially, (ii") implies (ii). □

Proof of Corollary 1.2: The implications (vi)⇒(ii) and (iv)⇒(vii) are trivial (take $\epsilon := 1/\| \begin{bmatrix} \tilde X & -\tilde Y \end{bmatrix} \|^2$). Implication (iv)⇒(vi) is from [19, Corollary 6.6] and (vii)⇒(iv') (see Theorem 2.1) holds for some suitable $\alpha \in \mathbb{D}$ by [7]. □

(In fact, (vii) is equivalent to (i)–(v) even if $\dim \mathtt{Y} = \infty$; it suffices that $\dim \mathtt{U} < \infty$.)

Next we need the following generalization of a classical result:

Lemma A.9 (R stabilizes P_{21}). *If R is stabilizing DPF-controller with internal loop for P, then R is a stabilizing controller with internal loop for P_{21}.*

(This was shown rigorously in [11, Lemma 7.3.5], but this can be observed from Figure 4.1: For $w = 0$ the closed-loop equations obviously define the map $(I - (P_{21})^R)^{-1}$ (see (3.1)) if we ignore the equation for z; thus, $(I - (P_{21})^R)^{-1}$ is contained in the "DPF closed-loop map" $(I - P^{R_{\mathrm{DPF}}})^{-1} \in \mathrm{H}^\infty$ for P and R.)

Proof of Theorems 4.1 and 4.2: 1° We first note that if R DPF-stabilizes P with internal loop, then it stabilizes P_{21}, by Lemma A.9, hence then P_{21} has a d.c.f. and R is equivalent to a canonical controller, by Theorem 1.1.

2° *(iii)⇒(iv):* Let $\tilde{X}^{-1}\tilde{Y}$ be a canonical controller that DPF-stabilizes P. Then $\tilde{X}^{-1}\tilde{Y}$ stabilizes P_{21}. One can verify that $\left[\begin{smallmatrix}\tilde{X} & 0\\ 0 & I\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix}0 & \tilde{Y}\\ 0 & 0\end{smallmatrix}\right]$ is a canonical controller that stabilizes P (see [11, Lemma 7.3.10] for details). By Lemma 3.5 it follows that there exists a r.c.f. $P = NM^{-1}$ such that

$$\begin{bmatrix} I & 0\\ 0 & I\end{bmatrix} = \begin{bmatrix}\tilde{X} & 0\\ 0 & I\end{bmatrix} M - \begin{bmatrix} 0 & \tilde{Y}\\ 0 & 0\end{bmatrix} N = \begin{bmatrix}\tilde{X}M_{11} - \tilde{Y}N_{21} & \tilde{X}M_{12} - \tilde{Y}N_{22}\\ M_{21} & M_{22}\end{bmatrix}, \tag{A.9}$$

in particular, $\tilde{X}M_{11} - \tilde{Y}N_{21} = I$, $M_{21} = 0$ and $M_{22} = I$, hence (iv) holds.

3° *(iv)⇒(iii):* Assume (iv). By Lemma 3.5 and (A.9), $\tilde{X}^{-1}\tilde{Y}$ DPF-stabilizes P iff we have

$$\begin{bmatrix}\tilde{X}M_{11} - \tilde{Y}N_{21} & \tilde{X}M_{12} - \tilde{Y}N_{22}\\ 0 & I\end{bmatrix} \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U}\times\mathtt{W}), \tag{A.10}$$

or equivalently, $\tilde{X}M_{11} - \tilde{Y}N_{21} \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U})$, or equivalently, iff the canonical controller $\tilde{X}^{-1}\tilde{Y}$ stabilizes $N_{21}M_{11}^{-1} = P_{21}$. By Theorem 1.1 (applied to P_{21}), such $\tilde{X}, \tilde{Y} \in \mathrm{H}^\infty$ do exist.

4° By duality, we get "(iii)⇔(v)", hence (iii)–(v) are equivalent. Moreover, in 3° we observed that if (iv) holds, then a canonical controller DPF-stabilizes P iff it stabilizes P_{21}. By Theorem 1.1, all such canonical controllers are given by the Youla parameterization (1.3) and at least one of them is strictly proper, hence (i) and (ii) are equivalent to (iii). Moreover, by Lemmata A.9 and 3.3, any DPF-stabilizing controller with internal loop for P is equivalent to some canonical controller. Thus, both theorems have been established. □

In a r.c.f. $P = NM^{-1}$, the inverse M^{-1} has the same maximal domain as P:

Lemma A.10. *If $P : \Omega \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ is holomorphic, where $\Omega \subset \mathbb{D}$ is connected, and $P = NM^{-1}$ is a α-r.c.f. for some $\alpha \in \Omega$, then $P = NM^{-1}$ is a z-r.c.f. for every $z \in \Omega$.*

(Indeed, if $\tilde{X}M - \tilde{Y}N = I$, then $I = (\tilde{X} - \tilde{Y}P)M = M(\tilde{X} - \tilde{Y}P)$ near α, hence on Ω.)

Thus, then any connected holomorphic extension of P (within $\mathbb{D}$) is a restriction of NM^{-1}.

Next we conclude the same with a "possibly non-connected Ω". E.g., if NM^{-1} has an α_1-r.c.f. and a α_2-r.c.f., then these are the same (modulo a unit):

Lemma A.11. *Let $\alpha_1, \alpha_2 \in \mathbb{D}$, $N \in \mathrm{H}^\infty(\mathtt{U}, \mathtt{Y})$, $M \in \mathrm{H}^\infty(\mathtt{U})$, $M(\alpha_1), M(\alpha_2) \in \mathcal{G}\mathcal{B}(\mathtt{U})$. Let $N_k M_k^{-1}$ be an α_k-r.c.f. of NM^{-1} ($k = 1, 2$). Then $\left[\begin{smallmatrix}N_2\\ M_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}N_1\\ M_1\end{smallmatrix}\right] V$ for some $V \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U})$; in particular, $N_1 M_1^{-1}$ is also an α_2-r.c.f. of NM^{-1}.*

Proof. Set $F := \left[\begin{smallmatrix}N\\ M\end{smallmatrix}\right]$, $F_k := \left[\begin{smallmatrix}N_k\\ M_k\end{smallmatrix}\right]$, and pick $G_k \in \mathrm{H}^\infty$ such that $G_k F_k = I$ ($k = 1, 2$). By Proposition A.1 (and Remark 7.1), we have $F = F_k V_k$ for some $V_k \in \mathrm{H}^\infty(\mathtt{U})$ such that $V_k(\alpha_k) \in \mathcal{G}\mathcal{B}(\mathtt{U})$ ($k = 1, 2$). Therefore $V_1 = G_1 F_2 V_2$ and $V_2 = G_2 F_1 V_1 = G_2 F_1 G_1 F_2 V_2$, hence $G_2 F_1 G_1 F_2 = I$ near α_2, hence on $\mathbb{D}$. Similarly, $TS = I$, where $T := G_1 F_2$, $S := G_2 F_1$. But $F_2 V_2 = F_1 V_1 = F_1 T V_2$, hence $F_2 = F_1 T$ near α_2, hence on $\mathbb{D}$. Set $V := T$. □

Proof of Lemma 6.1: 1° *"If":* Set $\begin{bmatrix}\tilde{X} & -\tilde{Y}\\ -\tilde{N} & \tilde{M}\end{bmatrix} := \left[\begin{smallmatrix}M & Y\\ N & X\end{smallmatrix}\right]^{-1}$. By (3.2) (and Remark 7.1 and Lemma 3.2), $\left[\begin{smallmatrix}I & -Q\\ -P & I\end{smallmatrix}\right]^{-1} = \begin{bmatrix}Y\tilde{N}+I & Y\tilde{M}\\ X\tilde{N} & X\tilde{M}\end{bmatrix} =: E$ on Ω.

2° *"Only if":* By (1.1), $N_0, M_0 \in \mathrm{H}^\infty$, where $N_0 := P(I - QP)^{-1}$ and $M_0 = (I - QP)^{-1}$. Since $P = N_0 M_0^{-1}$ on Ω, P has an α-r.c.f. and a β-r.c.f., by Lemma A.4 (cf. Remark 7.1). By Lemma A.11, any α-r.c.f. of P is a β-r.c.f. of P.

By Lemmata 3.3, 3.5 and 3.2, there exists an α-d.c.f. $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{smallmatrix}\right]^{-1}$ of P such that $X(\alpha) \in \mathcal{GB}(\mathtt{Y})$ and $Q = YX^{-1}$ on a neighborhood of α. By the above, $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$ is also a β-d.c.f. of P. Exchange the roles of P and Q to observe that $Q = YX^{-1}$ is also a β-r.c.f. (in particular, $X(\beta) \in \mathcal{GB}(\mathtt{Y})$).

3° By Proposition A.1 (cf. Remark 7.1), $\left[\begin{smallmatrix} M_1 & Y_1 \\ N_1 & X_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] \left[\begin{smallmatrix} V & 0 \\ 0 & V_Q \end{smallmatrix}\right]$ for some $V \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{U})$, $V_Q \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{Y})$, hence the claim on β-r.c.f.'s follows. The last claim now follows from Lemma A.10. □

Appendix B. Continuous-time proofs (for §7).

In this appendix we prove the results of §7.

Proof of Remark 7.1: This is straightforward, but we give some examples below. For clarity, in this proof we add the prefix "$(\mathbb{D}',\zeta)$-" when we refer to the redefined terminology of Remark 7.1; otherwise we refer to the original DT terminology of Sections 1–6.

By the Riemann Mapping Theorem, there exists a holomorphic function $f : \mathbb{D}' \to \mathbb{D}$ that has a holomorphic inverse and satisfies $f(\zeta) = 0$. Obviously, a function P' is $(\mathbb{D}',\zeta)$-proper (resp., $(\mathbb{D}',\zeta)$-H^∞) iff $P := P' \circ f^{-1}$ is proper (resp., H^∞). Moreover, a function Q is a proper stabilizing controller for P iff $Q' := Q \circ f$ is a $(\mathbb{D}',\zeta)$-proper stabilizing controller for $P \circ f = P'$ i.e., iff Q' is $(\mathbb{D}',\zeta)$-proper and $\left[\begin{smallmatrix} I & -Q' \\ -P' & I \end{smallmatrix}\right]^{-1}$ is a bounded holomorphic function on $\mathbb{D}'$ (because $\left[\begin{smallmatrix} I & -Q' \\ -P' & I \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & -Q \\ -P & I \end{smallmatrix}\right]^{-1} \circ f$ wherever either exists).

Similarly, also all other required old terminology is mapped to corresponding new $(\mathbb{D}',\zeta)$-terminology, hence all results mentioned in Remark 7.1 hold also for this new $(\mathbb{D}',\zeta)$-terminology.

E.g., if a ζ-proper function P' has a $(\mathbb{D}',\zeta)$-proper stabilizing controller, then $P := P' \circ f^{-1}$ has a r.c.f. $P = NM^{-1}$, by Theorem 1.1, hence than P' has the r.c.f. $N'(M')^{-1}$ with $N' := N \circ f,\ M' := M \circ f$. □

The original proofs of all DT results apply in their CT forms too except that part of the proof of Theorem 1.1 would require additional details. Also Remark 7.1 (with some additional work) could be used to prove almost all CT results, but in some cases it becomes more cumbersome than the use of the original proofs, so we shall in each proof below select the simplest method.

From now on we use the CT terminology defined in Definition 7.2 and below it unless we explicitly use the prefix *DT-* (when referring to the original DT terminology of Sections 1–6)) or the prefix ζ- (when referring to the terminology defined in Remark 7.1 for fixed $\zeta \in \mathbb{C}^+$ and $\mathbb{D}' := \mathbb{C}^+$). Thus, e.g., proper means H^∞_∞ but *DT-proper* (resp., ζ-*proper*) means holomorphic on a neighborhood of 0 (resp., ζ). Observe that in, e.g., ζ-*canonical controller* or ζ-H^∞ the prefix is redundant (but *DT*-H^∞ means holomorphic and bounded on $\mathbb{D}$, not on $\mathbb{C}^+$).

However, when referring to some result, we refer to its original form unless we use the prefix "CT-" or "ζ-". When we write, e.g., *CT-Corollary 3.6*, we refer to the CT-form of Corollary 3.6 that is established in Theorem 7.3 (its proof is given later below). Similarly, ζ-*Corollary 3.6* refers to the form established in Remark 7.1 (with $\mathbb{D}' = \mathbb{C}^+$).

We shall often need the fact that the (continuous-time) r.c.f.'s of a proper function are the same as its α-r.c.f.'s:

Lemma B.1 (α-r.c.f.). *Let $\omega \geq 0$, $P \in \mathrm{H}^\infty_\omega(\mathtt{U},\mathtt{Y})$ and $\alpha \in \mathbb{C}^+_\omega$. Then an α-r.c.f. of P is a r.c.f., and a r.c.f. of P is an α-r.c.f. A similar claim obviously holds for* *l.c.f.'s and d.c.f.'s.*

Proof. If $\tilde{X}, \tilde{Y}, M, N \in \mathrm{H}^\infty$, $I = \tilde{X}M - \tilde{Y}N$ (on $\mathbb{C}^+$) and $NM^{-1} = P$ on a neighborhood Ω of α, then $I = (\tilde{X} - \tilde{Y}P)M$ on $\mathbb{C}^+_\omega$ and $M(\alpha)^{-1}$ exists, hence then $I = M(\tilde{X} - \tilde{Y}P)$ on a neighborhood of α, hence on $\mathbb{C}^+_\omega$ too, so then $P = NM^{-1}$ is a (CT) r.c.f. The converse is analogous. □

Proof of Theorem 7.5: As in the proof of Lemma B.1, we observe that also here α-r.c.f.'s coincide with r.c.f.'s. The rest of Theorem 7.5 is already contained in ζ-Theorem 2.1 (for any fixed $\zeta \in \Omega_P$). □

Proof of Theorem 7.4 except CT-Lemma 3.7: Pick $\omega \geq 0$ such that $P \in \mathrm{H}^\infty_\omega$, set $\Omega_P := \mathbb{C}^+_\omega$ (and $\mathbb{D}' := \mathbb{C}^+$). Observe that a (CT) "canonical controller" is such also in terms of Remark 7.1 (when, e.g., $\zeta \in \mathbb{C}^+_\omega$).

1° The equivalence of (i)–(v) follows from Theorem 7.5 as follows: The equivalence "(iv)⇔(iv')" is contained in Theorem 7.5. For $\alpha = \zeta \in \mathbb{C}^+_\omega$, we get "(iv)⇔(v)" from "(iv)⇔(iv')" (twice) and ζ-Theorem 1.1 (P has a r.c.f. ⇔ it has a ζ-r.c.f. ⇔ $\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$ has a ζ-r.c.f. ⇔ $\left[\begin{smallmatrix} P & 0 \\ 0 & I \end{smallmatrix}\right]$ has a r.c.f.). Each of conditions (i) and (ii) is obviously equivalent to (ii') and/or (ii") (of Theorem 7.5). (iii') implies (iii); if (iii) holds, then P has a ζ-r.c.f. for some $\zeta \in \mathbb{C}^+_\omega$ (pick one in the definition of the controller), by ζ-Theorem 1.1(iii)&(iv), i.e., then (iv') holds (take $\alpha = \zeta$).

2° For any $\zeta \in \mathbb{C}^+_\omega$, we get the d.c.f. from ζ-Theorem 1.1 and Lemma B.1. Obviously, a canonical controller $\left[\begin{smallmatrix} 0 & Y \\ I & I-X \end{smallmatrix}\right]$ stabilizes P iff it ζ-stabilizes P (for some $\zeta \in \mathbb{C}^+_\omega$). A stabilizing controller $R \in \mathrm{H}^\infty_\infty$ with internal loop for P is a ζ-stabilizing controller with internal loop for P (for some $\zeta \in \mathbb{C}^+_\omega$), hence equivalent to a [ζ-]canonical controller, by ζ-Lemma 3.3, so the Youla parameterization claim holds (recall Lemma 7.7). □

Proof of Theorem 7.3: All propositions: The CT forms are given in [15].

Lemmata A.3, A.4 and 3.3: Use the original proofs.

Lemmata 3.4 and 3.5 and Corollaries 3.6 and 1.2: Use the original proofs, mutatis mutandis (see Definition 7.2).

Lemmata A.9, A.10, A.11, 3.1: These follow easily from Remark 7.1.

Lemma 3.7: Pick a d.c.f. $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] = \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix}^{-1} \in \mathrm{H}^\infty(\mathtt{U} \times \mathtt{Y})$ of P. Pick $\omega' > 0$ such that $\|\tilde{X}(s) - M^{-1}(s)\| = \|\tilde{Y}(s)P(s)\| < 1/2\|M\|_{\mathrm{H}^\infty}$ $\forall s \in \mathbb{C}^+_{\omega'}$ to have $\tilde{X}(s)^{-1}$ bounded on $\mathbb{C}^+_{\omega'}$. Thus, all canonical controllers are proper (see CT-Corollary 3.6). By CT-Lemmata 3.2 and 3.3, all stabilizing controllers with internal loop for P are proper.

Lemma 6.1: This follows easily from Remark 7.1 and Lemma B.1. □

Proof of Theorem 7.6: One easily verifies that $M(+\infty) \in \mathcal{GB}(\mathtt{U})$ [11, Proposition 6.3.1(c)]. Let $P = N_0 M_0^{-1}$ be a r.c.f. By CT-Proposition A.1, $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] = \left[\begin{smallmatrix} N_0 \\ M_0 \end{smallmatrix}\right] V$ for some $V \in \mathrm{H}^\infty$ such that V^{-1} is proper. Let $\left[\begin{smallmatrix} M_0 & Y \\ N_0 & X \end{smallmatrix}\right] \in \mathcal{G}\,\mathrm{H}^\infty(\mathtt{Y} \times \mathtt{U})$ (see Theorem 7.4) and set

$$\begin{bmatrix} M_0 & Y_0 \\ N_0 & X_0 \end{bmatrix} := \begin{bmatrix} M_0 & Y \\ N_0 & X \end{bmatrix} \begin{bmatrix} I & -VM(+\infty)^{-1}Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} M_0 & (1 - MM(+\infty)^{-1})Y \\ N_0 & X - NM(+\infty)^{-1}Y \end{bmatrix}. \tag{B.1}$$

Set $\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} := \left[\begin{smallmatrix} M_0 & Y_0 \\ N_0 & X_0 \end{smallmatrix}\right]^{-1} \in \mathcal{G}\,\mathrm{H}^\infty$. Since $Y_0(+\infty) = 0$, the function $\tilde{M}X_0 = I - \tilde{N}Y_0$ is boundedly invertible on some right half-plane, hence so is X_0 (because so is $\tilde{M}$). Thus, $Q = Y_0 X_0^{-1}$ is strictly proper (and stabilizing, by Theorem 7.4). □

Proof of Theorem 7.9: The equivalence of (ii) and (iii) follows from Theorem 7.4 (and the definitions). The rest follows from ζ-Theorem 4.2 and Lemma B.1 for a suitable ζ. □

Proof of Theorem 7.8: From ζ-Theorem 4.1 (for some ζ in the domain of P and R) we observe that the first equivalence and the last claim hold.

If Q is a stabilizing DPF-controller function for P then it is a [ζ-]stabilizing canonical controller for P_{21} given by the Youla parameterization (pick some ζ in the domain of Q), by ζ-Theorem 4.1. The same holds when Q is a stabilizing controller function for P_{21}, by Theorem 7.4. Therefore, the second equivalence in Theorem 7.8 follows from the first. □

Proof of Remark 7.10: In the proof of Remark 5.1, use $(\cdot - r)$ in place of $(r\cdot)$, for a suitable $r > 0$. □

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