BASS AND TOPOLOGICAL STABLE RANKS OF COMPLEX AND REAL ALGEBRAS OF MEASURES, FUNCTIONS AND SEQUENCES

KALLE MIKKOLA AND AMOL SASANE

Abstract. We compute the Bass stable rank and the topological stable rank of several convolution Banach algebras of complex measures on $(-\infty, \infty)$ or on $[0, \infty)$ consisting of a discrete measure and/or of an absolutely continuous measure.

We also compute the stable ranks of the convolution algebras $\ell^1(\mathbb{N}^n)$, $\ell^1(\mathbb{Z}^n)$, $\ell^1(S)$ and $\ell^1(S \cap \mathbb{R}_+)$, where S is an arbitrary subgroup of $\mathbb{R}$, of the almost periodic algebra AP and of AP $\cap H^\infty$, etc. We answer affirmatively the question posed by R. Mortini in [20].

For the above algebras, the polydisc algebra $A(\mathbb{D}^n)$, the algebra $C(\mathbb{T}^n)$ of continuous functions, and others, we also study their subsets (real Banach algebras) of real-valued measures, real-valued sequences or real-symmetric functions, and of corresponding exponentially stable algebras (for example, the *Callier–Desoer algebra* of causal exponentially decaying measures and L^1 functions), and we compute their stable ranks. Finally, we show that in some of these real algebras a variant of the *parity interlacing property* is equivalent to reducibility of a unimodular (or coprime) pair. Also corona theorems are presented and the existence of coprime fractions is studied; in particular, we list which of these algebras are Bézout domains.

Contents

Date: October 19, 2008.

2000 *Mathematics Subject Classification.* Primary 43A10; Secondary 19B10, 46J10, 32A38, 46J20, 93D25.

Key words and phrases. Bass stable range, topological stable rank, parity interlacing property, discrete measures, commensurate delays, almost periodic functions, absolutely summable sequences, real-valued functions, real-symmetric functions, Callier-Desoer algebra, polydisc algebra.

1. Introduction

The purpose of this article is to compute the Bass and topological stable ranks of various real and complex Banach algebras of measures, functions and sequences. En route we answer affirmatively the question posed by Raymond Mortini in [20], about the existence of a closed subalgebra $\mathcal{A}$ of H^∞ such that the maximal ideal ideal space $X(\mathcal{A})$ of $\mathcal{A}$ contains $\mathbb{D}$ as a dense subset, and such that the Bass stable rank of $\mathcal{A}$ is strictly greater than 1.

The notions of Bass stable rank and topological stable rank (often called "stable range") play important roles in algebraic and topological K-theories, respectively (see [3] and [23]), but they also have important applications in other areas, such as in the control-theoretic problem of the dynamic stabilization of fractional transfer functions (see [22] and [38]); this is explained briefly in Subsection 10.1.

We recall the definitions of Bass stable rank and topological stable ranks below.

Definition 1.1. Let $\mathcal{A}$ be a ring[1] with identity 1. Let $n \in \mathbb{N}$. An element $a \in \mathcal{A}^n$ is called *unimodular* ("a left-invertible vector") if there exists $b \in \mathcal{A}^n$ (a *left inverse*) such that

$$b \cdot a := \sum_{k=1}^{n} b_k a_k = 1.$$

We denote by $\mathrm{U}_n(\mathcal{A})$ the set of unimodular elements of $\mathcal{A}^n$. A unimodular $n+1$-tuple $a \in \mathrm{U}_{n+1}(\mathcal{A})$ is called *reducible* (or stable) if there exists $x \in \mathcal{A}^n$ such that

$$(a_1 + x_1 a_{n+1}, \dots, a_n + x_n a_{n+1}) \in \mathrm{U}_n(\mathcal{A}). \tag{1}$$

The *Bass stable rank* (denoted by $\operatorname{bsr}\mathcal{A}$) of $\mathcal{A}$ is the least integer $n \geq 1$ such that every $a \in \mathrm{U}_{n+1}(\mathcal{A})$ is reducible, and it is infinite if no such integer n exists.

Now let $\mathcal{A}$ denote a Banach algebra[2]. The *topological stable rank* (denoted by $\operatorname{tsr}\mathcal{A}$) of $\mathcal{A}$ is the least integer $n \geq 1$ such that $\mathrm{U}_n(\mathcal{A})$ is dense in $\mathcal{A}^n$, and it is infinite if no such integer exists.

Recall also that elements of U_2 are usually called (right) *coprime*. We have $\operatorname{bsr}\mathcal{A} \leq \operatorname{tsr}\mathcal{A}$ for every Banach algebra $\mathcal{A}$ [23]. Our main results on stable ranks are summarized in Table 1, but also some other algebras will be treated. Most of the notation used in Table 1 is introduced in Section 2 (particularly $\mathcal{M}_*^*$, AP and AP_+), but we explain some of it below.

By $\lfloor r \rfloor$ we denote the greatest integer $\leq r$ ($r \in \mathbb{R}$).

By $\ell^1(\mathbb{R})$ we denote the Banach algebra of absolutely summable functions $a : \mathbb{R} \to \mathbb{C}$ with $\|a\|_1 := \sum_{r\in\mathbb{R}} |a_r|$ and the convolution product defined by $(a * b)_r := \sum_{t\in\mathbb{R}} a_t b_{r-t}$. Its subalgebras $\ell^1(\mathbb{R}_+)$, $\ell^1(\mathbb{Z})$ and $\ell^1(\mathbb{N})$ have the same norm and the same product. Naturally, $a \in \ell^1(\mathbb{R})$ implies that $a_r = 0$ for all but countably many $r \in \mathbb{R}$.

We use the symbol $C(\mathbb{T}^n)$ to denote the Banach algebra of complex-valued continuous functions defined on $\mathbb{T}^n$, where $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, and we use the notation $A(\mathbb{D}^n)$ for the Banach algebra of continuous functions $\overline{\mathbb{D}}^n \to \mathbb{C}$ that are holomorphic on the *polydisc* $\mathbb{D}^n$, where $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$.

[1]By $(\mathcal{A}, +, \cdot)$ being a *ring* we mean that $(\mathcal{A}, +)$ is a group, $(\mathcal{A}, \cdot)$ is associative and unital, and $a(b+c) = ab + ac$, $(b+c)a = ba + ca$ $\forall a, b, c \in \mathcal{A}$. An algebra is a ring.

[2]By a Banach algebra we mean a real or complex Banach algebra with a unit element 1. We do not assume commutativity, but the algebras defined in this article happen to be commutative (except in the matrix-valued case).

Algebra $\mathcal{A}$	$\operatorname{bsr}\mathcal{A}$	$\operatorname{tsr}\mathcal{A}$	$\operatorname{bsr}\mathcal{A}_{\mathbb{R}}$	$\operatorname{tsr}\mathcal{A}_{\mathbb{R}}$
$\mathbb{C}\delta_0 + \mathrm{L}^1$	1	1	1	1
$\mathbb{C}\delta_0 + \mathrm{L}^1_+$	1	2	2	2
$\ell^1(\mathbb{Z}^n),\ \mathcal{M}^{(n)},\ \mathcal{M}^{(n)} + \mathrm{L}^1,\ C(\mathbb{T}^n)$	$\lfloor n/2 \rfloor + 1$	$\lfloor n/2 \rfloor + 1$	$\lfloor n/2 \rfloor + 1$	$\lfloor n/2 \rfloor + 1$
$\ell^1(\mathbb{N}^n),\ \mathcal{M}_+^{(n^+)},\ \mathcal{M}_+^{(n^+)} + \mathrm{L}^1_+,\ A(\mathbb{D}^n)$	$\lfloor n/2 \rfloor + 1$	$n+1$	$n+1$	$n+1$
$\ell^1(\mathbb{R}),\ \mathcal{M},\ \mathcal{M} + \mathrm{L}^1,\ \mathrm{AP}$	∞	∞	∞	∞
$\ell^1(\mathbb{R}_+),\ \mathcal{M}_+,\ \mathcal{M}_+ + \mathrm{L}^1_+,\ \mathrm{AP}_+$	∞	∞	∞	∞

TABLE 1. Stable ranks of algebras and their real-valued subalgebras

By $\mathcal{M}$ we mean the convolution Banach algebra of discrete (atomic) measures. It consists of the sums $\sum_{r\in\mathbb{R}} a_r\delta_r$ (with the total variation norm $\sum_{r\in\mathbb{R}} |a_r|$), where $a \in \ell^1(\mathbb{R})$ and $\delta_r \in \mathcal{M}$ is the unit mass at r. Thus $\mathcal{M}$ is isometrically isomorphic to $\ell^1(\mathbb{R})$. The convolution Banach algebras

$$\mathcal{M}_+ := \{\mu \in \mathcal{M} \colon \operatorname{supp}\mu \subset [0,\infty)\}, \tag{2}$$

$\mathcal{M} + \mathrm{L}^1 := \mathcal{M} + \mathrm{L}^1(\mathbb{R})$ (denoted by $\mathrm{LA}(-\infty,\infty)$ in [15, §§4.20]) and $\mathcal{M}_+ + \mathrm{L}^1_+ := \mathcal{M}_+ + \mathrm{L}^1(\mathbb{R}_+)$ (denoted by L(1)+A(1) in [15, §§4.19]) have also been considered previously in the literature, but we study their stable ranks here. We present these and some other subalgebras of $\mathcal{M} + \mathrm{L}^1$ in further detail in Section 2. These other subalgebras include the algebra $\mathcal{M}^{(1)}$ of measures with *commensurate delays* (for example, $a_r = 0$ for $r \notin \mathbb{Z}$) and the algebra $\mathcal{M}_+^{(n^+)}$ (respectively, $\mathcal{M}^{(n)}$) generated by n independent (over $\mathbb{Z}$) δ_r's (respectively, δ_r's and δ_{-r}'s).

When $\mathcal{A}$ stands for some of such "$\mathcal{M}_*^*$" algebras, then the notation $\mathcal{A}_{\mathbb{R}}$ in Table 1 represents the subset of real-valued elements of $\mathcal{A}$, which is a *real* Banach algebra. If $\mathcal{A} = \ell^1(\mathbb{N})$, then $\mathcal{A}_{\mathbb{R}} := \ell^1(\mathbb{N};\mathbb{R})$ is the subset of $\mathcal{A}$ consisting of real-valued absolutely summable sequences $\mathbb{N} \to \mathbb{R}$; similar notation is used when $\mathbb{N}$ is replaced by $\mathbb{Z}$, $\mathbb{R}_+$ and $\mathbb{R}$.

When $\mathcal{A}$ stands for $C(\mathbb{T}^n)$, $A(\mathbb{D}^n)$, AP or $\mathrm{AP}_+ := \mathrm{AP} \cap H^\infty$, then by

$$\mathcal{A}_{\mathbb{R}} := \{f \in \mathcal{A} : \forall z,\ f(\overline{z}) = \overline{f(z)}\}$$

we denote its subset of *real-symmetric* functions. The real Banach algebra $A(\mathbb{D}^n)_{\mathbb{R}}$ coincides with the subset (of $A(\mathbb{D}^n)$) of functions whose Taylor series at the origin have real coefficients.

One easily verifies that a complex Borel measure is real-valued iff its Laplace transform is real-symmetric (on $i\mathbb{R}$, hence wherever the transform converges absolutely) and that an element of $\ell^1(\mathbb{N}^n)$ or $\ell^1(\mathbb{Z}^n)$ is real-valued iff its Z-transform is real-symmetric (on $\mathbb{T}^n$, hence wherever the transform converges absolutely); further details are given in Section 2.

These real Banach algebras are often more important than the complex ones, because in most physical applications the data is real and only real solutions are usable.

The facts that $\operatorname{bsr}\ell^1(\mathbb{N}) = 1$ and $\operatorname{bsr}\ell^1(\mathbb{Z}) = \operatorname{tsr}\ell^1(\mathbb{Z}) = 1$ were already known (see [29] and [10], respectively), and we have the isometries $\ell^1(\mathbb{Z}) \approx \mathcal{M}^{(1)}$ and $\ell^1(\mathbb{N}) \approx \mathcal{M}_+^{(1^+)}$. Also $\operatorname{bsr} C(\mathbb{T}^n) = \lfloor n/2 \rfloor + 1$ and $\operatorname{tsr} C(\mathbb{T}^n) = \lfloor n/2 \rfloor + 1$ were known; even better, the facts that $\operatorname{bsr} C(X;\mathbb{C}) = \lfloor n/2 \rfloor + 1$ and $\operatorname{bsr} C(X;\mathbb{R}) = n + 1$, where $n := \dim(X)$, were shown in [37], and (if X is compact) $\operatorname{tsr} C(X;\mathbb{C}) = \lfloor n/2 \rfloor + 1$ in [23]. (Note that, for example, $C(\mathbb{T}^n)_{\mathbb{R}} \neq C(\mathbb{T}^n;\mathbb{R})$.) It was also known that $\operatorname{bsr} A(\mathbb{D}^n) = \lfloor n/2 \rfloor + 1$ [8] (case $n = 1$ in [17]), $\operatorname{tsr} A(\mathbb{D}^n) = n + 1$ [7], $\operatorname{bsr} A(\mathbb{D})_{\mathbb{R}} = \operatorname{tsr} A(\mathbb{D})_{\mathbb{R}} = 2$ [31], and $\operatorname{bsr}\mathrm{AP} = \operatorname{tsr}\mathrm{AP} = \infty$ [33]. The result $\operatorname{bsr} \ell^1(\mathbb{N}^n) = \lfloor n/2 \rfloor + 1$ will be based on [8], and the result $\operatorname{bsr} \mathbb{C}\delta_0 + \mathrm{L}^1_+ = 1$ will be reduced to [29]. All other results in Table 1 seem to be completely new.

We note that $\operatorname{bsr} H^\infty = 1$ [36], and $\operatorname{tsr} H^\infty = 2$ [34]. Sergei Treil had conjectured that $\operatorname{bsr} H^\infty_{\mathbb{R}} = 2$, and it has recently been proved by Raymond Mortini and Brett Wick that $\operatorname{bsr} H^\infty_{\mathbb{R}} = \operatorname{tsr} H^\infty = 2$; see [21]. In Lemma 6.2 we construct, for any given n, an unimodular vector in U_n of $\mathcal{FM}_{+\mathbb{R}} \subset (\mathrm{AP}_+)_{\mathbb{R}} \subset H^\infty_{\mathbb{R}}$ that is not reducible in AP.

The following is our main result.

Theorem 1.2. *All results presented in Table 1 hold true (for any $n = 1, 2, 3, \ldots$).*

(The proof is given immediately after Corollary 7.9 below.)

Note, in particular, that $\operatorname{bsr} \mathcal{A} = \operatorname{bsr} \mathcal{A}_{\mathbb{R}} = \operatorname{tsr} \mathcal{A} = \operatorname{tsr} \mathcal{A}_{\mathbb{R}} = 1$ when $\mathcal{A}$ equals $\ell^1(\mathbb{Z})$, $\mathbb{C}\delta_0 + \mathrm{L}^1$, $\mathcal{M}^{(1)}$, $\mathcal{M}^{(1)} + \mathrm{L}^1$ or $C(\mathbb{T})$, and that $\operatorname{bsr} \mathcal{A} = 1$ when $\mathcal{A}$ equals $\ell^1(\mathbb{N})$, $\mathbb{C}\delta_0 + \mathrm{L}^1_+$, $\mathcal{M}^{(1^+)}_+$, $\mathcal{M}^{(1^+)}_+ + \mathrm{L}^1_+$ or $A(\mathbb{D})$. Also the corresponding *exponentially stable* subalgebras have $\mathrm{bsr} = 1$, as shown in Theorem 9.4. An example of these is

$$\ell^{1,\exp}(\mathbb{N}) := \{a \in \ell^1(\mathbb{N})\colon \sum_{k=0}^{\infty} r^k |a_k| < \infty \text{ for some } r > 1\}, \tag{3}$$

and another one is $\mathcal{M}^{1,\exp}_+ = \{\sum_{k=0}^\infty a_k \delta_{kT} \colon a \in \ell^{1,\exp}(\mathbb{N})\}$. Such algebras have been popular in control theory at least since the introduction of the Callier–Desoer algebra [4]. In §9 we show that all results in Table 1 and many others hold for the corresponding exponential algebras as well.

Unfortunately, the real variants (for example, $\ell^1(\mathbb{N};\mathbb{R})$ and $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$) of all "causal" complex algebras mentioned in the above paragraph have $\mathrm{bsr} = 2$. The reducible elements of U_2 of these real algebras are characterized by the *parity interlacing property* (§10). This means that if $(f, g) \in \mathrm{U}_2(\mathcal{A})$, then there exists $h \in \mathcal{A}$ such that $f + hg$ is invertible iff $\widehat{f}$ has the same sign at each real zero of $\widehat{g}$. That this holds for $\mathcal{A}$ equal to $\mathcal{F}^{-1} A(\mathbb{D})_{\mathbb{R}}$ was shown in [41], but we show this, among others, for $\ell^1(\mathbb{N};\mathbb{R})$, $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$ and for the corresponding exponential subalgebras; see §10. In control theory this reducibility is equivalent to $\widehat{g}/\widehat{f}$ being stabilizable by a stable controller (one can use $-\hat{h}$) [38].

The article is organized as follows. The notation is presented in §2. In §3 we recall a few results on general relations between stable ranks of Banach algebras. In §4 (and in §7) we establish corona theorems and other results with necessary and/or sufficient conditions for unimodularity in the algebras treated in this article.

The finite entries in Table 1, except those for algebras of the form $\mathcal{M}^*_* + \mathrm{L}^1_*$, are proved in §5. There we also prove fairly general stable rank results for subalgebras of $C(\mathbb{D}^n)$ and of $C(\mathbb{T}^n)$ and establish a powerful tool for constructing nonreducible unimodular continuous functions.

In §6 we show that $\operatorname{bsr} \mathcal{A} \geq \lfloor n/2 \rfloor + 1$ whenever $\mathcal{A}$ is a subalgebra of $\mathcal{M}$ or of $\mathcal{M}_{\mathbb{R}}$ and $\mathcal{A}$ contains $(\mathcal{M}^{(n)})_{\mathbb{R}}$. This implies all the $\mathrm{bsr} = \mathrm{tsr} = \infty$ results in Table 1 and answers the question posed by R. Mortini in [20].

In §7 we treat algebras of the form $\mathcal{M}^*_* + \mathrm{L}^1_*$ including those listed in Table 1.

In §8 we study the stable ranks and coronas of $\ell^1(S)$ and of $\ell^1(S \cap \mathbb{R}_+)$, where S is an additive subgroup of $\mathbb{R}$. Obviously, $\ell^1(S)$ is isomorphic to the algebra $\mathcal{M}^{(S)}$ of discrete measures on S. Naturally, also $\mathcal{M}^{(S)} + \mathrm{L}^1(\mathbb{R})$ and $\mathcal{M}^{S \cap \mathbb{R}_+} + \mathrm{L}^1(\mathbb{R}_+)$ are treated. As a corollary of the above, we obtain results for the algebra $\ell^1(E)$ (isomorphic to $\mathcal{M}^{(n)} \cap \mathcal{M}_+$), where $E := \{\alpha \in \mathbb{Z}^n \colon \sum_{k=1}^n \alpha_k T_k \geq 0\}$; here n and the $\mathbb{Q}$-independent $T_1, T_2, \ldots, T_n > 0$ are arbitrary. Also AP^S and AP^S_+, the sup-norm-closures of $\mathcal{FM}^{(S)}$ and $\mathcal{FM}^{S\cap\mathbb{R}_+}$, are studied. For AP^S_+ and $\mathcal{M}^{(S)}_+$, the corona theorems were given already in [24].

Stable ranks, unimodularity and other properties of "exponentially stable" measure, function or sequence algebras are studied in §9.

As mentioned below Theorem 1.2, the parity interlacing property and reducible coprime pairs in many real algebras are treated in §10. In Subsection 10.1 we explain the control-theoretic relevance of reducibility and stable ranks, in strong stabilization, simultaneous stabilization etc.

In Subsection 10.2 we explain when a fraction g/f, where $f, g \in \mathcal{A}$, $f \neq 0$, equals a coprime fraction $\tilde{g}/\tilde{f}$, where $(\tilde{f}, \tilde{g}) \in \mathrm{U}_2(\mathcal{A})$; in particular, we observe which of the algebras treated in this article are *Bézout domains*. Analogous results hold for matrix-valued functions too.

When treating a complex algebra we always treat also the corresponding real algebra of real-valued measures or sequences (or of real-symmetric functions (transformations)).

Acknowledgement: The authors would like to thank Raymond Mortini for giving us a proof of Lemma 3.9.(1).

2. Notation

In this section we present most of our notation and terminology (including $C(A;B)$, L^1, ℓ^1, I, $\mathcal{M}^*_*$, $*$, $\mathcal{F}$, AP, AP_+ and "causal"). See

- §1 for U_n, $b \cdot a$, bsr, tsr, $\lfloor n/2 \rfloor \leq n/2$, ℓ^1, $*$, $C(\mathbb{T}^n)$, $A(\mathbb{D}^n)$, $\cdot_\mathbb{R}$ (including $\mathcal{A}_\mathbb{R}$, $(\mathrm{AP})_\mathbb{R}$, $\mathrm{L}^1(\mathbb{R}_+)_\mathbb{R} = \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$ etc.), "ring", "unimodular", "reducible", "real-symmetric" and "coprime";
- §3 for "algebra", "subalgebra", "topological algebra", "morphism", "full", and "ideal";
- §4 for "maximal ideal space"="$X(\mathcal{A})$", "corona", and "symmetrization";
- §5 for $\bar{B}_r(a)$, "(topological) function algebra", and "Cayley transform".
- §8 for $\mathcal{M}^{(S)}$, $\mathcal{M}^{(S)}_+$, $\mathcal{M}^{(n)}_+$, AP^S, AP^S_+ etc. and $\dim_\mathbb{Q} S$;
- §9 for $\mathrm{A}^{\exp}$, $(\mathcal{M} + \mathrm{L}^1)^{\exp}$, $\ell^{1,\exp}$ etc.

Note: we do not distinguish between row and column vectors.

Here we define some symbols.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$	$\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$, $\mathbb{Q} = \{$rational numbers$\}$,
$\mathbb{K}$	either $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$,
$\mathbb{R}_+$	$[0, \infty)$,
$\mathbb{T}$, $\mathbb{D}$, $\mathbb{C}_+$	$\mathbb{T} := \{z \in \mathbb{C} \colon \lvert z\rvert = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} \colon \lvert z\rvert < 1\}$, $\mathbb{C}_+ := \{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$.
$\mathcal{M}^*_*$	some of the various measure algebras defined below.
$\mathcal{F}$, $\widehat{\ }$	the Laplace/Fourier or Z-transform (see later below).
δ_a	the unit mass at a (when $a \in \mathbb{R}$). Note that $(\delta_a * f)(t) = f(t - a)$.
m_k	the k-dimensional Lebesgue measure m.
z_k	the kth coordinate of z if, for example, $z \in \overline{\mathbb{D}}^n$ (then $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ for every $\alpha \in \mathbb{Z}^n$, and $\mathrm{e}^{tz} := (\mathrm{e}^{tz_1}, \mathrm{e}^{tz_2}, \dots, \mathrm{e}^{tz_n}) \in \mathbb{C}^n$ for every $t \in \mathbb{R}$, etc.).
T_k	We assume that $T_1, T_2, T_3, \dots \in (0, \infty)$ are fixed and linearly independent over $\mathbb{Q}$ (or equivalently, over $\mathbb{Z}$).

T $\quad T := (T_1, T_2, \ldots, T_n)$ for some n. Sometimes $T = T_1 > 0$ (when $n = 1$).

We use the notation $C(A;B)$ for the set of continuous functions $A \to B$; and $C(A) := C(A;\mathbb{C})$. Analogously, $\mathrm{L}^1(A) := \mathrm{L}^1(A;\mathbb{C})$, $\ell^1(A) := \ell^1(A;\mathbb{C})$. By n and k we denote arbitrary elements of $\{1,2,3,\ldots\}$ unless otherwise specified. By 1 we denote the unit element of an algebra and by $I := \mathrm{diag}(1,1,\ldots,1)$ the identity matrix. A *polynomial* means a (real or complex, depending on the context) linear combination of the functions z^n ($n \in \mathbb{N}$) or a k-tuple $(p_1, p_2, \ldots, p_k)$ of such polynomials (for any $k = 1, 2, \ldots$).

We will consider the following Banach algebras with the operations of addition and convolution (with δ_0 as the unit element):

$\mathcal{M}$ $\quad$ discrete (or atomic) measures on $\mathbb{R}$,
$f = \sum_{n\in\mathbb{Z}} a_n \delta_{t_n}$, $a = (a_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, $\cdots < t_{-1} < 0 = t_0 < t_1 < \ldots$.

$\mathcal{M}_+$ $\quad$ causal discrete measures,
$f = \sum_{n\in\mathbb{N}} a_n \delta_{t_n}$, $a \in \ell^1(\mathbb{N})$, $0 = t_0 < t_1 < t_2 < \ldots$.

$\mathbb{C}\delta_0 + \mathrm{L}^1$ $\quad$ absolutely continuous measures with identity,
$f = f_\mathrm{a} + a\delta_0$, $a \in \mathbb{C}$, $f_\mathrm{a} \in \mathrm{L}^1(\mathbb{R})$.

$\mathbb{C}\delta_0 + \mathrm{L}^1_+$ $\quad$ causal absolutely continuous with identity
$f = f_\mathrm{a} + a\delta_0$, $a \in \mathbb{C}$, $f_\mathrm{a} \in \mathrm{L}^1(\mathbb{R}_+)$.

$\mathcal{M} + \mathrm{L}^1$ $\quad$ measures without continuous singular part
$f = f_\mathrm{a} + \sum_{n\in\mathbb{Z}} a_n \delta_{t_n}$, $f_\mathrm{a} \in \mathrm{L}^1(\mathbb{R})$, $a \in \ell^1(\mathbb{Z})$, $\cdots < t_{-1} < 0 = t_0 < t_1 < \ldots$.

$\mathcal{M} + \mathrm{L}^1$ is equipped with the norm $\|f\| := \|f_\mathrm{a}\|_{\mathrm{L}^1(\mathbb{R})} + \|a\|_{\ell^1(\mathbb{Z})}$. The same norm is used for each of the above algebras and for the ones to follow. They are all closed subalgebras of $\mathcal{M} + \mathrm{L}^1$.

$\mathcal{M}_+ + \mathrm{L}^1_+$ $\quad$ causal measures without continuous singular part
$f = f_\mathrm{a} + \sum_{n\in\mathbb{N}} a_n \delta_{t_n}$, $f_\mathrm{a} \in \mathrm{L}^1(\mathbb{R}_+)$, $a \in \ell^1(\mathbb{N})$, $0 = t_0 < t_1 < t_2 < \ldots$.

In the following, let $T > 0$.

$\mathcal{M}^{(1)}$ $\quad$ periodic discrete measures,
$f = \sum_{n\in\mathbb{Z}} a_n \delta_{nT}$, $a \in \ell^1(\mathbb{Z})$.

$\mathcal{M}^{(1^+)}_+$ $\quad$ causal periodic discrete measures,
$f = \sum_{n\in\mathbb{N}} a_n \delta_{nT}$, $a \in \ell^1(\mathbb{N})$.

$\mathcal{M}^{(1)} + \mathrm{L}^1$ $\quad$ absolutely continuous plus periodic discrete measures,
$f = f_\mathrm{a} + \sum_{n\in\mathbb{Z}} a_n \delta_{nT}$, $f_\mathrm{a} \in \mathrm{L}^1(\mathbb{R})$, $a \in \ell^1(\mathbb{Z})$, $\cdots < 0 = t_0 < t_1 < \ldots$.

$\mathcal{M}^{(1^+)}_+ + \mathrm{L}^1_+$ $\quad$ the causal elements of $\mathcal{M}^{(1)} + \mathrm{L}^1$,
$f = f_\mathrm{a} + \sum_{n\in\mathbb{N}} a_n \delta_{nT}$, $f_\mathrm{a} \in \mathrm{L}^1(\mathbb{R}_+)$, $a \in \ell^1(\mathbb{N})$, $0 = t_0 < t_1 < t_2 < \ldots$.

$\mathcal{M}^{(n)}, \mathcal{M}^{(n^+)}_+$ $\quad$ see below Remark 2.1.

$\mathcal{A}^n$ $\quad$ The set of n-tuples $(a_1, a_2, \ldots, a_n)$ of the elements of $\mathcal{A}$, respectively. Naturally, $\mathcal{M}^n$ is an example of this.

If $\mathcal{M}^*_*$ denotes one of the classes above, then by $\mathcal{F}\mathcal{M}^*_*$ we denote the set of Laplace transforms of elements of $\mathcal{M}^*_*$, where

$$(\mathcal{F}f)(s) := \widehat{f}(s) := \int_\mathbb{R} \mathrm{e}^{-st} f(t)\, dt.$$

With the operations of pointwise addition and multiplication, and equipped with the same (coinduced) norm as $\mathcal{M}_*^*$, the set $\mathcal{F}\mathcal{M}_*^*$ becomes a Banach algebra that is isometrically isomorphic to $\mathcal{M}_*^*$; in particular, the stable ranks are the same. We recall that $\mathcal{F}(f * g) = \widehat{f}\,\widehat{g}$ for all $f, g \in \mathcal{M} + \mathrm{L}^1$ and that $\mathcal{F}\delta_r = \mathrm{e}^{-r\cdot}$ $(r \in \mathbb{R})$.

For $\ell^1(\mathbb{Z})$ or $\ell^1(\mathbb{N})$ the symbol $\mathcal{F}$ stands for the *Z-transform*:

$$(\mathcal{F}a)(z) := \widehat{a}(z) := \sum_k a_k z^k.$$

Similarly, if $a \in \ell^1(\mathbb{Z}^n)$ (respectively, $\ell^1(\mathbb{N}^n)$), then

$$\widehat{a}(z_1, \ldots, z_n) := \sum_{(k_1,\ldots,k_n)\in\mathbb{Z}^n} a_k z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}.$$

Observe that also here $\mathcal{F}(a * b) = \widehat{a}\,\widehat{b}$ on $\mathbb{T}^n$ (respectively, on $\overline{\mathbb{D}}^n$), where

$$(a * b)_k := \sum_{j\in\mathbb{Z}^n} a_j b_{k-j} \quad (k \in \mathbb{Z}^n).$$

For $\mathcal{A}$ equal to $\mathcal{M} + \mathrm{L}^1$ (respectively, $\mathcal{M}_+ + \mathrm{L}^1_+$, $\ell^1(\mathbb{Z}^n)$, $\ell^1(\mathbb{N}^n)$) and $f \in \mathcal{A}$, the function $\widehat{f}$ is uniformly continuous on $i\mathbb{R}$ (respectively, $\overline{\mathbb{C}_+}$, $\mathbb{T}^n$, $\overline{\mathbb{D}}^n$) and $\sup|\widehat{f}| \le \|f\|_{\mathcal{A}}$.

We identify $\mathcal{F} : \mathcal{A} \to \mathcal{F}\mathcal{A}$ with the corresponding function $\mathcal{F} : \mathcal{A}^n \to (\mathcal{F}\mathcal{A})^n$ (analogously for other operations and sets).

Remark 2.1. The algebras $\ell^1(\mathbb{N})$, $\ell^1(\mathbb{Z})$, $\ell^1(\mathbb{R}_+)$ and $\ell^1(\mathbb{R})$ are isometrically isomorphic (as Banach algebras) to $\mathcal{M}_+^{(1^+)}$, $\mathcal{M}^{(1)}$, $\mathcal{M}_+$ and $\mathcal{M}$, respectively. Similarly, $\ell^1(\mathbb{Z}^n) \approx \mathcal{M}^{(n)}$ and $\ell^1(\mathbb{N}^n) \approx \mathcal{M}_+^{(n^+)}$, as explained below. Analogous claims hold for the corresponding real-valued subsets (real Banach algebras): $\ell^1(\mathbb{N};\mathbb{R}) \approx (\mathcal{M}_+^{(1^+)})_{\mathbb{R}}$, etc.

In the above isometries $a \in \ell^1(\mathbb{R})$ (or $a \in \ell^1(\mathbb{R}_+)$) is identified with $\sum_{r\in\mathbb{R}} a_r\delta_r \in \mathcal{M}$ but $a \in \ell^1(\mathbb{Z})$ (or $a \in \ell^1(\mathbb{N})$) with $\sum_{n\in\mathbb{Z}} a_n\delta_{nT} \in \mathcal{M}^{(1)}$.

Next we explain how the Banach algebras $\ell^1(\mathbb{N}^n)$ and $\ell^1(\mathbb{Z}^n)$ are isometrically isomorphic to the algebras $\mathcal{M}_+^{(n^+)}$ and $\mathcal{M}^{(n)}$ of n noncommensurate delays.

Let $n \in \{1, 2, 3, \ldots\}$. Let $T_1, T_2, T_3, \ldots > 0$ be linearly independent over $\mathbb{Q}$. Then the smallest closed subalgebra $\mathcal{M}_+^{(n^+)}$ of $\mathcal{M}$ containing $\delta_{T_1}, \delta_{T_2}, \ldots, \delta_{T_n}$ is obviously isometrically isomorphic to $\ell^1(\mathbb{N}^n)$ through

$$\ell^1(\mathbb{N}^n) \ni a \mapsto \sum_{\alpha\in\mathbb{N}^n} a_\alpha \delta_T^\alpha, \tag{4}$$

where $\delta_T^\alpha := \delta_{T_1}^{\alpha_1}\delta_{T_2}^{\alpha_2}\cdots\delta_{T_n}^{\alpha_n} = \delta_{\alpha_1T_1+\alpha_2T_2+\cdots+\alpha_nT_n} = \delta_{\alpha\cdot T}$. Similarly, the smallest closed subalgebra $\mathcal{M}^{(n)}$ of $\mathcal{M}$ containing $\delta_{\pm T_1}, \delta_{\pm T_2}, \ldots, \delta_{\pm T_n}$ is isometrically isomorphic to $\ell^1(\mathbb{Z}^n)$.

By AP (respectively, AP_+) we denote the closure of $\mathcal{F}\mathcal{M}$ (respectively, $\mathcal{F}\mathcal{M}_+$) with respect to the supremum norm on the imaginary axis $i\mathbb{R}$. Note that AP is the algebra of *almost periodic* functions $i\mathbb{R} \to \mathbb{C}$ and that $\mathrm{AP}_+ := \mathrm{AP} \cap H^\infty$ consists of those elements that have a holomorphic extension to $\{z \in \mathbb{C}\colon \operatorname{Re} z \ge 0\}$.

A measure is called *causal* if its support lies on $\mathbb{R}_+$. An element of $\ell^1(\mathbb{Z}^n)$ is called *causal* if its support lies on $\mathbb{N}^n$. In these cases causality is obviously equivalent to the Laplace (respectively, Z-) transformation being holomorphic and bounded on $\mathbb{C}_+$ (respectively, on

$\mathbb{D}$). Therefore, we often call also $A(\mathbb{D}^n)$ and AP_+ (and $\ell^1(\mathbb{N}^n)$, $\ell^1(\mathbb{R}_+)$, $\mathcal{M}_+$ etc.) *causal* algebras and $C(\mathbb{T}^n)$ and AP (and $\ell^1(\mathbb{Z}^n)$, $\ell^1(\mathbb{R})$, $\mathcal{M}$ etc.) *noncausal* algebras.

If $a \in \mathcal{A}^n$ and $b \in \mathcal{A}$, then we often write $(a, b) := (a_1, \ldots, a_n, b) \in \mathcal{A}^{n+1}$. Thus, the condition $\mathrm{bsr}\,\mathcal{A} \le n$ can be rewritten as "if $(a, b) \in \mathrm{U}_{n+1}(\mathcal{A})$ and $b \in \mathcal{A}$, then there exists $x \in \mathcal{A}^n$ such that $a + xb \in \mathrm{U}_n(\mathcal{A})$" (by Lemma 3.1).

3. Preliminaries on stable ranks

We assume that every *algebra* (a real/complex vector space with a bilinear multiplication) is associative, and has a unit. We say that $\mathcal{A} \subset \mathcal{A}'$ is a *subalgebra* of $\mathcal{A}'$ if $\mathcal{A}$ and $\mathcal{A}'$ are algebras (with same operations and scalar field) and have the same unit. A *topological algebra* means a topological vector space which is an algebra and where also the multiplication is continuous. Obviously, also the definition of tsr is meaningful for any topological algebra. A *Banach algebra* $\mathcal{A}$ is an algebra that is also a Banach space and satisfies $\|ab\| \le \|a\|\|b\|$ $(a, b \in \mathcal{A})$. An algebra *morphism* means a linear, multiplicative ($f(ab) = f(a)f(b)$) mapping.

We sometimes use without further mention the following lemma [37, Theorem 1] [23, p. 303]. It says that if either stable rank condition holds for some n, then it holds for every bigger n too.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra. If every $a \in \mathrm{U}_{n+1}(\mathcal{A})$ is reducible, then so is every $a \in \mathrm{U}_{n+2}(\mathcal{A})$. If $\mathrm{U}_n(\mathcal{A})$ is dense in $\mathcal{A}^n$, then $\mathrm{U}_{n+1}(\mathcal{A})$ is dense in $\mathcal{A}^{n+1}$.*

We recall the following from [23].

Proposition 3.2. *For every Banach algebra $\mathcal{A}$ we have* $\mathrm{tsr}\,\mathcal{A} \ge \mathrm{bsr}\,\mathcal{A}$.

(If $\mathcal{A}$ is noncommutative, then one can define left and right tsr's and bsr's, but Proposition 3.2 holds for any of them.)

Many of our tsr results are constructive, and we sketch simpler constructive proofs for some special cases in the notes to §5. We remark that they all lead to constructive bsr results.

Remark 3.3. If $f, g, x, y \in \mathcal{A}$ are such that $xf + yg = 1$, and $\mathrm{tsr}\,\mathcal{A} = 1$, then there exists $w \in \mathrm{U}_1(\mathcal{A})$ so close to x that $wf + yg \in \mathrm{U}_1(\mathcal{A})$, that is, that $f + hg \in \mathrm{U}_1(\mathcal{A})$, where $h := w^{-1}y \in \mathcal{A}$. So any constructive $\mathrm{tsr}\,\mathcal{A} = 1$ result is also a constructive $\mathrm{bsr}\,\mathcal{A} = 1$ result. Analogously, any constructive $\mathrm{tsr}\,\mathcal{A} = n$ result is also a constructive $\mathrm{bsr}\,\mathcal{A} = n$ result, as one observes from the proof (in [23] or in [6]) of Proposition 3.2.

Note also that the tsr and bsr results are robust to small errors (in, e.g., f, g, x, y, w and h) in the sense described in Subsection 10.1.

In [8, p. 545] it is conjectured that $\mathrm{tsr}\,\mathcal{A} \le 2\,\mathrm{bsr}\,\mathcal{A}$ for every complex Banach algebra.

Lemma 3.4. *If $\mathcal{A}$ is a Banach algebra, then $\mathrm{U}_n(\mathcal{A})$ is open.*

Proof. Let $f \in \mathrm{U}_n(\mathcal{A})$ and $g \cdot f = 1$ for some $g \in \mathcal{A}^n$. If $h \in \mathcal{A}^n$ is sufficiently close to f, then $g \cdot h$ is invertible and hence $h \in \mathrm{U}_n(\mathcal{A})$, because $(g \cdot h)^{-1} g \cdot h = 1$. □

If $\mathcal{A}$ is a subalgebra of a unital algebra $\mathcal{A}'$ and every $\mathcal{A}'$-invertible $a \in \mathcal{A}$ has $a^{-1} \in \mathcal{A}$, then $\mathcal{A}$ is called *full*. An equivalent condition is that $\mathrm{U}_1(\mathcal{A}) = \mathcal{A} \cap \mathrm{U}_1(\mathcal{A}')$.

Lemma 3.5. *If $\mathcal{A}$ is a dense and full subalgebra of a Banach algebra $\mathcal{A}'$, then $\mathcal{A}^n \cap \mathrm{U}_n(\mathcal{A}') = \mathrm{U}_n(\mathcal{A})$ $(n \in \{1, 2, 3, \ldots\})$.*

Proof. If $f \in \mathrm{U}_n(\mathcal{A})$, then $f \in \mathcal{A} \cap \mathrm{U}_n(\mathcal{A}')$, so assume that $f \in \mathcal{A} \cap \mathrm{U}_n(\mathcal{A}')$ and hence $g \cdot f = 1$ for some $g \in (\mathcal{A}')^n$. Pick $h \in \mathcal{A}^n$ so close to g that $h \cdot f$ is invertible (in $\mathcal{A}'$, hence in $\mathcal{A}$). Then $(h \cdot f)^{-1} h \in \mathcal{A}$ and $(h \cdot f)^{-1} h \cdot f = 1$, hence $f \in \mathrm{U}_n(\mathcal{A})$. □

The following is a special case of [35, Theorem 2.2].

Lemma 3.6. *If $\mathcal{A}$ is a dense and full subalgebra of a Banach algebra $\mathcal{A}'$, then* $\operatorname{bsr} \mathcal{A} \leq \operatorname{bsr} \mathcal{A}'$.

(The fullness assumption can be weakened. Moreover, if $\mathcal{A}'$ is complex and commutative, then $\operatorname{bsr} \mathcal{A}' \leq \operatorname{bsr} \mathcal{A} + 2$, by [8, Corollary 3.18]. Swan's open problem asks if $\operatorname{bsr} \mathcal{A}' = \operatorname{bsr} \mathcal{A}$.)

We quote the following from Badea [2, Proposition 4.12] (its proof applies in the real case as well).

Proposition 3.7. *Let $\mathcal{A}$ and $\mathcal{A}'$ be topological algebras and let $f : \mathcal{A} \to \mathcal{A}'$ be a continuous algebra morphism with a dense image. Then* $\operatorname{tsr} \mathcal{A} \geq \operatorname{tsr} \mathcal{A}'$.

Corollary 3.8. *Let $\mathcal{A}$ be a topological algebra and let $\mathcal{A}'$ be a Banach algebra. If $f : \mathcal{A} \to \mathcal{A}'$ is a continuous injective algebra morphism with dense and full image $f[\mathcal{A}] \subset \mathcal{A}'$, then* $\operatorname{tsr} \mathcal{A} \geq \operatorname{tsr} \mathcal{A}' \geq \operatorname{bsr} \mathcal{A}' \geq \operatorname{bsr} \mathcal{A}$*; if, in addition, f^{-1} is continuous, then* $\operatorname{tsr} \mathcal{A} = \operatorname{tsr} \mathcal{A}'$.

(Much of this is given in [6, Theorem 3 and Corollary 3] and also the other results are known at least to some extent. Many of the claims actually hold for fairly general topological rings too.)

Proof. The chain of inequalities follows from Lemma 3.6 and Propositions 3.7 and 3.2. Assume then that $n := \operatorname{tsr} \mathcal{A}' < \infty$, $a \in f[\mathcal{A}^n]$ and $\epsilon > 0$. Pick $a' \in \mathrm{U}_n(\mathcal{A}')$ with $\|a - a'\| < \epsilon/2$ and then $b \in f[\mathcal{A}]^n$ such that $\|a' - b\| < \epsilon/2$ is so small that $b \in \mathrm{U}_n(\mathcal{A}')$. By Lemma 3.5, $b \in \mathrm{U}_n(f(\mathcal{A})] = f[\mathrm{U}_n(\mathcal{A})]$, so $\operatorname{tsr} f[\mathcal{A}] \leq n$. If f^{-1} is continuous, then $\operatorname{tsr} \mathcal{A} = \operatorname{tsr} f[\mathcal{A}]$. □

Note that $\mathcal{M} + \mathrm{L}^1$ and its subalgebras are complex and commutative. We also have

$$\mathcal{M} * \mathrm{L}^1(\mathbb{R}) \subset \mathrm{L}^1(\mathbb{R}) \text{ and } \mathcal{M}_+ * \mathrm{L}^1(\mathbb{R}_+) \subset \mathrm{L}^1(\mathbb{R}_+)$$

(these facts will be used without further mention). In fact $\mathrm{L}^1(\mathbb{R})$ is an closed 2-sided ideal of $\mathcal{M} + \mathrm{L}^1(\mathbb{R})$. Recall that a subgroup $(J, +)$ is a *2-sided ideal* of a ring R if $JR = RJ = J$. Since the quotient algebra $(\mathcal{M} + \mathrm{L}^1)/\mathrm{L}^1$ is isometrically isomorphic to $\mathbb{C}\delta_0 + \mathrm{L}^1$, the following lemma says that, for example, $\operatorname{bsr}(\mathcal{M} + \mathrm{L}^1) \geq \operatorname{bsr}(\mathbb{C}\delta_0 + \mathrm{L}^1)$.

Lemma 3.9. *Let $\mathcal{A}$ be a commutative $\mathbb{K}$-Banach algebra and let J be an ideal of $\mathcal{A}$ with $\mathbb{K}J \subset J$.*

(1) *We have* $\operatorname{bsr} \mathcal{A} \geq \max\{\operatorname{bsr} \mathcal{A}/J, \operatorname{bsr}(\mathbb{K}1 + J)\}$.
(2) *If J is closed, then* $\operatorname{tsr} \mathcal{A} \geq \operatorname{tsr} \mathcal{A}/J$.

Proof. (1) The inequality $\operatorname{bsr} \mathcal{A} \geq \operatorname{bsr} \mathcal{A}/J$ is from [37, Theorem 4]. The inequality $\operatorname{bsr} \mathcal{A} \geq \operatorname{bsr}(\mathbb{K}1 + J)$ was proved by Raymond Mortini, and we include his proof here:

To this end, suppose that $\mathcal{A}$ has the stable rank n. We want to show that $\mathbb{K}1 + J$ has stable rank less than or equal to n. Let $(f_1, \cdots, f_n, h)$ be an invertible tuple in $\mathbb{K}1 + J$. Consider the following two possible cases:

1° If $f_{j_0} = a + F$ for some $F \in J$ and $a \in \mathbb{K}$, $a \neq 0$, then $I := (f_1, \cdots, f_n, Fh)$ is also an invertible tuple in $\mathbb{K}1 + J$. In fact, suppose that the ideal I is contained in a maximal ideal M of $\mathbb{K}1 + J$. Since M is prime, either F or h is in M. But, by our hypothesis, h can't be in M; so F is in M. But then $a = f_{j_0} - F \in M$; a contradiction.

Since $\mathcal{A}$ has the stable rank n, there exist $x_j \in A$ such that $(f_1 + x_1Fh, \cdots, f_n + x_nFh)$ is an invertible tuple in $\mathcal{A}^n$. But $x_jF \in J \subset \mathbb{K}1 + J$; hence the tuple $(f_1, \cdots, f_n, h)$ is reducible in $\mathbb{K}1 + J$.

2° If all the f_j are in J, then necessarily $h \notin J$. Hence $(f_1 + h, f_2, \cdots, f_n, h)$ is an invertible tuple in $(\mathbb{K}1 + J)^{n+1}$. Note that $f_1 + h \notin J$; so we have the situation of the first case (for $j_0 = 1$). Thus there are $y_j \in \mathbb{K}1 + J$ such that $(f_1 + h + y_1h, f_2 + y_2h, \cdots, f_n + y_nh) = (f_1 + (y_1 + 1)h, f_2 + y_2h, \cdots, f_n + y_nh)$ is an invertible tuple in $(\mathbb{K}1 + J)^n$. Hence $(f_1, \cdots, f_n, h)$ is reducible in $\mathbb{K}1 + J$.

This completes the proof.

(2) This is proved in [23, Theorem 4.3].

□

Next we observe that if $(a, b) \in \mathrm{U}_2$, then $(a + cb, b) \in \mathrm{U}_2$, and if $(a + cb, b)$ is reducible, then so is (a, b).

Lemma 3.10. *Let $\mathcal{A}$ be a ring. Let $a \in \mathcal{A}^n$ and $b \in \mathcal{A}$ be "coprime", that is, $(a, b) \in \mathrm{U}_{n+1}(\mathcal{A})$. Let $c \in \mathcal{A}^n$ and $w \in \mathrm{U}_1(\mathcal{A})$. Then $(a + cb, b) \in \mathrm{U}_{n+1}$ and $(wa, b) \in \mathrm{U}_{n+1}$. If $(w^{-1}(a + cb), b)$ is reducible, then so is (a, b).*

(Above w and w^{-1} are interchangeable if $\mathcal{A}$ is commutative.)

Proof. If $x \cdot a + y \cdot b = 1$, then $x \cdot (a + cb) + (y - x \cdot c)b = 1$ and $xw^{-1} \cdot (wa) + y \cdot b = 1$. If $w^{-1}(a + cb) + hb \in \mathrm{U}_n$, then $\mathrm{U}_n \ni (a + cb) + whb = a + (c + wh)b$. □

4. Unimodularity in our algebras

In this section we present corona theorems and other results on unimodularity. Further similar results are presented in §7.

First we note that a unimodular measure in some $\mathcal{M}_*^*$ algebra has a unimodular discrete part.

Lemma 4.1. *Let $\mathcal{A}$ be equal to $\mathcal{M} + \mathrm{L}^1$ or $\mathcal{M}_+ + \mathrm{L}^1_+$. If $\mu + f \in \mathrm{U}_n(\mathcal{A})$ (with $f \in (\mathrm{L}^1)^n$ and $\mu \in \mathcal{M}^n$), then $\mu \in \mathrm{U}_n(\mathcal{A})$.*

Proof. Assume that $1 = (\nu + g) * (\mu + f) = \nu * \mu + h$ (where $a * b := \sum_{k=1}^n a_k * b_k$), where $h := \nu * f + g * \mu + f * g \in \mathrm{L}^1$, hence $h = 0$ and $\nu * \mu = 1$. □

In the above proof we showed that the discrete part ν of the inverse of $\mu + f$ is a left inverse of μ. In Lemma 7.1 we shall establish the nontrivial converse: if ν is any left inverse of μ, then there exists $g \in \mathrm{L}^1$ such that $\nu + g$ is a left inverse of $\mu + f$.

If $\mathcal{A}$ is complex and commutative, then by $X(\mathcal{A})$ we denote the *maximal ideal space* of $\mathcal{A}$, that is, the set of nonzero homomorphisms $\mathcal{A} \to \mathbb{C}$ with weak* topology. Recall that $X(\mathcal{A}) \subset \mathcal{A}^*$ and that $\|\Lambda\| = 1$ for every $\Lambda \in X(\mathcal{A})$ [27].

It is well known that $f \in \mathcal{A}^n$ is in $\mathrm{U}_n(\mathcal{A})$ iff $\Lambda f \neq 0 \ \forall \Lambda \in X(\mathcal{A})$, that is, iff $\epsilon := \inf_{\Lambda \in X(\mathcal{A})} |\Lambda f| > 0$. For the latter condition, $X(\mathcal{A})$ can obviously be replaced by a dense subset. We state here also the converse (which is essentially proved on [11, pp. 202–203]).

Lemma 4.2 ($X(\mathcal{A})$)**.** *Assume that $\mathcal{A}$ is a complex commutative Banach algebra. Let $X_0 \subset X(\mathcal{A})$. Then the following are equivalent.*

(i) *X_0 is dense in $X(\mathcal{A})$.*
(ii) *Given any $n \geq 1$ and $f \in \mathcal{A}$, we have $f \in \mathrm{U}_n(\mathcal{A})$ iff there exists $\epsilon > 0$ such that $|\Lambda f_1| + \cdots + |\Lambda f_n| \geq \epsilon$ for every $\Lambda \in X_0$.*

The set $X(\mathcal{A}) \setminus \overline{X_0}$ is called the "corona" of the maximal ideal space. A "corona theorem" states that the corona is empty, that is, it is a special case of Lemma 4.2. In Theorem 4.3 below we present below corona theorems (or dense subsets of maximal ideal spaces) for most algebras used in this article.

This theorem shows that the point evaluations of the Fourier or Z-transform of a measure or ℓ^1 function, respectively, form a dense subset of the maximal ideal case in the cases treated in (a) and the whole maximal ideal space in (b).

Theorem 4.3 (Corona Theorem)**.** *Let $n, k \in \{1, 2, \cdots\}$.*

(a1) *Let $\mathcal{A}$ be any one of $\mathcal{M} + \mathrm{L}^1$, $\mathcal{M}^{(1)} + \mathrm{L}^1$, $\mathcal{M}$, $\mathcal{M}^{(1)}$, $\mathbb{C}\delta_0 + \mathrm{L}^1$ and $\mathcal{F}^{-1}\,\mathrm{AP}$. Then $f \in \mathcal{F}\mathcal{A}^n$ is unimodular ($f \in \mathrm{U}_n(\mathcal{F}\mathcal{A})$) iff there exists $\epsilon > 0$ such that $|f(z)| \geq \epsilon$ for every $z \in i\mathbb{R}$.*
(a2) *Part (a1) above also holds if we replace $\mathcal{A}$ by $\mathcal{A}_+$ (respectively, by $\mathcal{M}^{(n)}$) and $i\mathbb{R}$ by $\mathbb{C}_+$ (respectively, by $i\mathbb{R}$).*
(b) *Let $\mathcal{A}$ stand for the algebra $\mathcal{M}^{(1)}$ (respectively, $\mathcal{M}_+^{(1^+)}$, $\mathbb{C}\delta_0 + \mathrm{L}^1$, $\mathbb{C}\delta_0 + \mathrm{L}^1_+$, $\ell^1(\mathbb{Z}^k)$, $\ell^1(\mathbb{N}^k)$, $C(\mathbb{T}^k)$, $A(\mathbb{D}^k)$). Then $f \in \mathcal{F}\mathcal{A}^n$ is unimodular iff $f(z) \neq 0$ for every z in $i\mathbb{R} \cup \{\infty\}$ (respectively, $\overline{\mathbb{C}_+} \cup \{\infty\}$, $i\mathbb{R} \cup \{\infty\}$, $\overline{\mathbb{C}_+} \cup \{\infty\}$, $\mathbb{T}^k$, $\overline{\mathbb{D}}^k$, $\mathbb{T}^k$, $\overline{\mathbb{D}}^k$).*

(Here $|\cdot|$ denotes any norm on $\mathbb{C}^n$, and "$f(z) \neq 0$" means that "$f(z) \neq (0, 0, \ldots, 0)$". Naturally, by $(\mathcal{M} + \mathrm{L}^1)^+$ we refer to $\mathcal{M}_+ + \mathrm{L}^1_+$, etc. We have $\mathcal{M}^{(n)}$ in (a2), because $\mathcal{M}_+^{(n)}$ will not be defined before §8.)

Proof. We first prove (b):

1° The maximal ideal spaces of $\mathbb{C}\delta_0 + \mathrm{L}^1$, $\mathbb{C}\delta_0 + \mathrm{L}^1_+$, $\ell^1(\mathbb{Z}^k)$, $\ell^1(\mathbb{N}^k)$, $C(\mathbb{T}^n)$ and $A(\mathbb{D}^n)$ equal the sets mentioned in (b), by [13, pp. 107 and 112] and [27, p. 271] (or [38, pp. 338–339]). This proves (b) for those classes, by Lemma 4.2.

2° The algebras $\mathcal{M}^{(1)}$ and $\mathcal{M}_+^{(1^+)}$: We reduce these to $\ell^1(\mathbb{Z})$ and to $\ell^1(\mathbb{N})$, respectively. We have $\widehat{g}(s) := \sum_n a_n \exp(-nTs) = \sum_n a_n z^n = \widehat{a}(z)$, where $z = \exp(-Ts)$; this mapping $g \mapsto a \in \ell^1$ is an isometric isomorphism. If the components of f are of this form, say $\widehat{f}_k(s) = \widehat{b}_k(z)$ for $k = 1, \ldots, n$, then we have $|\widehat{f}(s)| = |\widehat{b}(z)|$ (for any s with $z := \exp(-Ts)$), so then $|\widehat{f}| \geq \epsilon$ on $i\mathbb{R}$ (respectively, on $\mathbb{C}_+$) iff $|\widehat{b}| \geq \epsilon$ on $\mathbb{T}$ (respectively, on $\mathbb{D}$). By 1°, an equivalent condition is that $b \in \mathrm{U}_n(\ell^1(\mathbb{Z}))$ (respectively, $b \in \mathrm{U}_n(\ell^1(\mathbb{N}))$), which in turn holds iff $f \in \mathrm{U}_n(\mathcal{M}^{(1)})$ (respectively, $f \in \mathrm{U}_n(\mathcal{M}_+^{(1^+)})$), by the isomorphism.

(a) For the classes mentioned in (b) the equivalence follows by continuity and compactness. For the others it suffices to show that $i\mathbb{R}$ (respectively, $\mathbb{C}_+$) is dense in the maximal ideal space of $\mathcal{A}$ (respectively, of $\mathcal{A}_+$), by Lemma 4.2.

1° The algebras $\mathcal{M} + \mathrm{L}^1$, $\mathcal{M}_+ + \mathrm{L}^1_+$, $\mathcal{M}_+$, AP, AP_+: The proofs for density can be found from the following references: $\mathcal{M} + \mathrm{L}^1$ in [15, Theorem 4.20.4], $\mathcal{M}_+ + \mathrm{L}^1_+$ in [38, p. 342], AP in [12], and AP_+ and $\mathcal{M}_+$ in [24].

2° The algebra $\mathcal{M}$: If $f \in \mathcal{F}(\mathcal{M})^n$ satisfies $|f| \geq \epsilon > 0$ on $i\mathbb{R}$, then $g \cdot f = 1$ for some $g \in \mathcal{F}(\mathcal{M} + \mathrm{L}^1)$. Obviously, the discrete part of g is a left inverse of f.

3° The algebras $\mathcal{M}^{(1)} + \mathrm{L}^1$ and $\mathcal{M}_+^{(1^+)} + \mathrm{L}_+^1$: From Lemma 4.1 and Proposition 8.2 (whose proof is self-contained) we conclude that if $\mu, \nu \in \mathcal{M}^n$, $f, g \in \mathrm{L}^1(\mathbb{R})^n$ and $(\nu + g) \cdot (\mu + f) = \delta_0$, then $(\nu_S + g) \cdot (\mu + f) = \delta_0$ (where $S = \mathbb{Z}$). Thus, the result for $\mathcal{M}^{(1)} + \mathrm{L}^1$ follows from that for $\mathcal{M} + \mathrm{L}^1$. Analogously, the result for $\mathcal{M}_+^{(1^+)} + \mathrm{L}_+^1$ follows from that for $\mathcal{M}_+ + \mathrm{L}_+^1$.

4° The algebra $\mathcal{M}^{(n)}$: The proof is similar to 3° above.

(Part of our proofs are from [19, Theorem 4.1.6], which contains further results.) □

For further algebras of the form $\mathcal{A} \subset \mathcal{M}$ or $\mathcal{A} + \mathrm{L}_*^1$, corona theorems are given in Lemma 7.3 and in Corollary 7.4. (For $\mathcal{A}$ equal to $\mathcal{M}^{(S)}$, $\mathcal{M}_+^{(S)}$ or $\mathcal{M}_+^{(n)}$, see Proposition 8.2, where also $\ell^1(S)$ and $\ell^1(S \cap \mathbb{R}_+)$ are covered.) Theorem 4.5 (respectively, 9.3) is the analogous corona theorem for real-valued (respectively, for "exponentially stable") measures or sequences. For $\mathrm{AP}^S \subset \mathrm{AP}$ and $\mathrm{AP}_+^S \subset \mathrm{AP}_+$, corona theorems are given in Corollary 8.3. For $\ell^1(\mathbb{R})$, $\ell^1(\mathbb{R}_+)$ or $\mathcal{M}_+^{(n^+)}$, use Remark 2.1.

We note that if $\mathcal{A} \subset \mathcal{M} + \mathrm{L}^1$, then $\mathcal{A}_\mathbb{R}$ equals the subset of real-valued elements of $\mathcal{A}$ (the elements $f_\mathrm{a} + \sum_n a_n \delta_{t_n} \in \mathcal{A}$ for which f_a is real-valued and $a_n \in \mathbb{R}\ \forall n$). Equivalently, then $\mathcal{A}_\mathbb{R} = \{\mu \in \mathcal{A} \colon \widehat{\mu}(\overline{z}) = \overline{\widehat{\mu}(z)}\ \forall z\} = \mathcal{A} \cap (\mathcal{M} + \mathrm{L}^1)_\mathbb{R}$.

The elements of these and our other algebras can be made real-valued or real-symmetric by *symmetrization*.

Lemma 4.4 (symmetrization). *If $\mu \in \mathcal{M} + \mathrm{L}^1$, then $\operatorname{Re} \mu = (\mu + \bar{\mu})/2 \in (\mathcal{M} + \mathrm{L}^1)_\mathbb{R}$. Moreover, $\hat{\bar{\mu}}(s) = \overline{\widehat{\mu}(\bar{s})}$ for all $s \in i\mathbb{R}$ (in fact for all $s \in \mathbb{C}_+$ if $\mu \in \mathcal{M}_+ + \mathrm{L}_+^1$).*

Let $\mathcal{A}$ stand for any of the algebras mentioned in Theorem 4.3. If $f \in \mathcal{F}\mathcal{A}$, then $f_R := (f + \overline{f(\bar{\cdot})})/2 \in \mathcal{F}\mathcal{A}_\mathbb{R}$, and $\|f_R\|_\infty \le \|f\|_\infty$. If $f, g \in \mathcal{F}\mathcal{A}^n$, $g \cdot f = 1$ and $f \in \mathcal{F}\mathcal{A}_\mathbb{R}^n$, then $g_R \in \mathcal{F}\mathcal{A}_\mathbb{R}^n$ and $g_R \cdot f = 1$.

Proof. Most of this is obvious. For the last claim, set $g_c(z) := \overline{g(\overline{z})}$, $g_R := (g + g_c)/2$. Then $f_c = f = f_R$, hence

$$g_R \cdot f = \frac{g + g_c}{2} \cdot f = \frac{g \cdot f + g_c \cdot f_c}{2} = \frac{1 + 1_c}{2} = 1.$$

□

This makes it easy to extend Theorem 4.3 to the real algebras.

Theorem 4.5 (Real Corona Theorem). *Theorem 4.3 holds even if we replace $\mathcal{A}$ by $\mathcal{A}_\mathbb{R}$ (in the "iff" claims). Thus $\mathrm{U}_n(\mathcal{A}_\mathbb{R}) = \mathcal{A}_\mathbb{R}^n \cap \mathrm{U}_n(\mathcal{A})$.*

Proof. By the last claim of Lemma 4.4, we have $\mathcal{A}_\mathbb{R}^n \cap \mathrm{U}_n(\mathcal{A}) \subset \mathrm{U}_n(\mathcal{A}_\mathbb{R})$. The converse is trivial. The former claim of Theorem 4.5 follows from the latter and Theorem 4.3. □

5. Finitely many generators: $\mathcal{M}^{(n)}$, $\ell^1(\mathbb{N}^n)$, $A(\mathbb{D}^n)$, $C(\mathbb{T}^n)$, ...

In this section we study $\mathcal{M}_+^{(n^+)}$, $\ell^1(\mathbb{Z}^n)$, $A(\mathbb{D}^n)$, $C(\mathbb{T}^n)$ and other finitely generated algebras (and the corresponding real algebras). In the technical Subsection 5.1 we construct some examples of nonreducible elements of U_n. In Subsection 5.2 we compute the stable ranks of many algebras, partially based on the examples in Subsection 5.1.

For the causal real algebras studied here we have bsr ≥ 2. However, in many of them a (coprime) pair $(f, g) \in \mathrm{U}_2$ is reducible iff it has the "parity interlacing property", as we shall see in §10.

5.1. **Nonreducible unimodular vectors.** In this subsection we present examples that will be used to prove most of our results of the form "bsr $\geq$". We start with an auxiliary result followed by a powerful tool for constructing nonreducible functions.

In the following lemma we observe that the identity function on a nice set $E \subset \mathbb{R}^n$ can be continuously extended to $\mathbb{R}^n$ so that its values remain in $\bar{E}$.

Lemma 5.1. *Let $E \subset \mathbb{R}^n$ be convex, open and bounded. Then there exists $b \in C(\mathbb{R}^n; \bar{E})$ such that $b(x) = x$ $(x \in E)$.*

Proof. Without loss of generality, we can assume that $0 \in E$. By [16, p. 82], the Minkowski functional $N(x) := \inf\{t > 0\colon x/t \in E\}$ is a norm on $\mathbb{R}^n$. Because every norm on $\mathbb{R}^n$ is continuous, the function $b'(x) := x/N(x)$ is a continuous function $\mathbb{R}^n \setminus \{0\} \to \partial E$; obviously, $b'(x) \in \partial E$ $(x \in \mathbb{R}^n \setminus \{0\})$. It is a simple exercise to show that $b'(x) = x$ $(x \in \partial E)$. Let $b(x) = x$ for $x \in E$ and $b(x) := b'(x)$ elsewhere. □

A standard tool in many stable rank proofs in the literature is Brouwer's fixed point principle [28] or the stability of the origin for the identity mapping on $\mathbb{R}^n$ [37]. We now formulate this method explicitly and extend it by allowing the "boundary function" F below to be nonzero. Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and that E° denotes the interior of E.

Lemma 5.2. *Let $E \subset \mathbb{K}^n$ be compact, $f_j, g \in C(E; \mathbb{K})$ $(j \leq n)$ and $g(x) = 0$ $(x \in \partial E)$. Set $f = (f_1, \ldots, f_n)$. If the function $F(x) := x - f(x)$ $(x \in \partial E)$ has an extension $F \in C(\mathbb{K}^n \setminus E^\circ; E)$, then for every $h \in C(E; \mathbb{K}^n)$ there exists $x \in E$ such that $(f + gh)(x) = 0$.*

For this continuous extendibility of F, a sufficient condition is obviously that $F|_{\partial E}$ is a constant that belongs to E. A necessary condition is that $F[\partial E] \subset E$; this condition is also sufficient if there exists a retraction $b : \mathbb{K}^n \setminus E^\circ \to \partial E$ (this means that b is continuous and $b(x) = x$ for every $x \in \partial E$).

By Lemma 5.1, the retraction exists if E is the closure of a convex, open and bounded set.

Proof. Define

$$G(x) := \begin{cases} x - f(x) - g(x)h(x), & \text{if } x \in E; \\ F(x), & \text{if } x \notin E \end{cases}. \tag{5}$$

Since $g = 0$ on ∂E, we have $F = G$ on ∂E, and so $G \in C(\mathbb{K}^n; \mathbb{K}^n)$. G is bounded since E is compact. By Brouwer's fixed point theorem (Lemma A.3), there exists $x \in \mathbb{K}^n$ such that $G(x) = x$. But if $x \notin E$, then $x = G(x) = F(x) \in E$, a contradiction. So we must have $x \in E$. Consequently, $x = G(x) = x - (f + gh)(x)$, that is, $(f + gh)(x) = 0$.

The second paragraph is straightforward (for example, $F \circ b$ is an extension of F if b is a retraction). □

If $a \in \mathbb{R}^n$, then $\bar{B}_r(a) := \{x \in \mathbb{R}^n\colon |x - a| \leq r\} \subset \mathbb{R}^n \subset \mathbb{C}^n$. We now give a "unimodular" real-coefficient polynomial $(f, g) : \mathbb{C}^n \to \mathbb{C}^{n+1}$ that is not reducible in any subalgebra of $C(\mathbb{D}^n)_\mathbb{R}$.

Corollary 5.3. *Let $0 \neq n \in \mathbb{N}$, $a \in \mathbb{R}^n$ and $r > 0$. Define the real-coefficient polynomials $g, f_j \in C(\mathbb{C}^n; \mathbb{C})_\mathbb{R}$ by*

$$g(z) := r^2 - \sum_{j=1}^{n} (z_j - a_j)^2, \quad f_j(z) := z_j - a_j \;\; (j = 1, \ldots, n,\; z \in \mathbb{C}^n). \tag{6}$$

If $h \in C(\bar{B}_r(a);\mathbb{R}^n)$*, then* $(f+gh)(x) = 0$ *for some* $x \in \bar{B}_r(a)$*. Moreover,* $(f,g) : \mathbb{C}^n \to \mathbb{C}^{n+1}$ *has no zeros and* $f_1f_1 + f_2f_2 + \cdots + f_nf_n + r^{-2}g = 1$*.*

Note that if $a_j = 1/2\ \forall j$*,* $r = 1/3$*, then* $E := \bar{B}_r(a) \subset [1/6, 5/6]^n$*, hence then for every* $h \in C(\mathbb{D}^n)^n_{\mathbb{R}}$ *there exists* $x \in E \subset \mathbb{D}^n$ *such that* $(f+hg)(x) = 0$*.*

This follows from Lemma 5.2 with $\mathbb{K} := \mathbb{R}$, $E := \bar{B}_r(a) \subset \mathbb{K}^n$, because here $x - f(x) = a \in E$ $(x \in \partial E)$.

All typical algebras of real-symmetric functions on $\mathbb{D}^n$ have bsr $\geq n+1$.

Corollary 5.4. *Let* $\mathcal{A}$ *be a subalgebra of* $C(\mathbb{D}^n)_{\mathbb{R}}$*. If* $\mathcal{A}$ *contains the polynomials, then* $\operatorname{bsr}\mathcal{A} \geq n+1$*.*

Indeed, now with f, g from Corollary 5.3 we have $(f,g) \in \mathrm{U}_{n+1}(\mathcal{A})$ but $f + hg \notin \mathrm{U}_n(\mathcal{A})$ for every $h \in \mathcal{A}^n$.

A converse is given in Theorem 5.16.

By a *function algebra* we mean an algebra of functions with the standard operations $(fg)(z) := f(z)g(z)$, $(f+g)(z) = f(z) + g(z)$, $(\alpha f)(z) := \alpha f(z)$. A *topological function algebra* means a topological algebra that is a function algebra.

Corollary 5.5. *Let* $E \subset \mathbb{R}^n$ *have a nonempty interior. Assume that* $\mathcal{A} \subset C(E;\mathbb{R})$ *is a function algebra, and that* $\mathcal{A}$ *contains the polynomials. Then* $\operatorname{bsr}\mathcal{A} \geq n+1$*.*

Indeed, the f and g defined by (6) satisfy $(f,g) \in \mathrm{U}_{n+1}(\mathcal{A})$ and (f,g) is not reducible.

We note that in [37] it was shown that $\operatorname{bsr}\mathcal{A} \geq n+1$ and $\operatorname{bsr} C(E;\mathbb{R}) = n+1$ also if, for example, E is any topological space with $d(E) = n$ and $\mathcal{A}$ is a dense subring of the set of bounded real-valued continuous functions on E.

The three above corollaries are useful for real algebras only, so next we construct fairly elementary elements of $C(\mathbb{T}^m;\mathbb{C})$ that are not reducible. These will also be used in certain proofs later on.

Corollary 5.6. *Let* $0 \neq n \in \mathbb{N}$*,* $r \in (0,1)$*. Set* $\mathbb{T}_r := \{\mathrm{e}^{it} : t \in [-r, \pi/2 + r]\} \subset \mathbb{T}$*. Define* $g, f_j \in C(\mathbb{T}^{2n};\mathbb{C})_{\mathbb{R}}$ *by*

$$f_j(z) := [z_{2j-1} + z_{2j-1}^{-1} + z_{2j} - z_{2j}^{-1}]/2 = \operatorname{Re} z_{2j-1} + i \operatorname{Im} z_{2j},$$

$$g(z) := r^2 - |f(z)|^2 = r^2 - \sum_{j=1}^{n} f_j(z) f_j(\bar{z}) \tag{7}$$

$(j = 1, \ldots, n,\ z \in \mathbb{T}^n)$*. If* $h \in C(\mathbb{T}_r^{2n};\mathbb{C}^n)$*, then* $(f+gh)(z) = 0$ *for some* $z \in \mathbb{T}_r^{2n}$*. Moreover,* $(f,g) : \mathbb{T}^{2n} \to \mathbb{C}^{n+1}$ *has no zeros, and* $f_1f_1 + f_2f_2 + \cdots + f_nf_n + r^{-2}g = 1$*.*

Proof. Note that f and g are real-coefficient polynomials in z_j and $\bar{z}_j = z_j^{-1}$ $(j \leq 2n)$, and that so is $\bar{f}_j = f_j(\bar{\cdot})$, hence (7) holds. In particular, $f_j, g \in C(\mathbb{T}^{2n};\mathbb{C})_{\mathbb{R}}$ $(j \leq n)$. If $f(z) = 0$, then $g(z) = r^2$, and so (f,g) has no zeros. Set

$$a := (\pi/2, \ldots, \pi/2) \in \mathbb{C}^n, \quad E := \{w \in \mathbb{C}^n : |w - a| \leq r\}.$$

If $w \in E$, then $\operatorname{Re} w_j \in [\pi/2 - r, \pi/2 + r]$ and $\operatorname{Im} w_j \in [-r, r]$. Define $\phi \in C(E;\mathbb{T}_r^{2n})$ by

$$\phi(w)_{2j-1} := \mathrm{e}^{i\cos^{-1}(\operatorname{Re} w_j - \pi/2)}, \quad \phi(w)_{2j} := \mathrm{e}^{i\sin^{-1}\operatorname{Im} w_j} \quad (w \in E,\ j \leq n).$$

For $w \in E$, $z := \phi(w) \in \mathbb{T}_r^{2n}$, we have

$$\operatorname{Re} f(z)_j = \operatorname{Re} z_{2j-1} = \operatorname{Re} w_j - \pi/2, \quad \operatorname{Im} f_j(z) = \operatorname{Im} z_{2j} = \operatorname{Im} w_j \quad (w \in E),$$

hence $f(\phi(w)) = w - a$ $(w \in E)$. Consequently, $g(\phi(w)) = r^2 - |w-a|^2$, and so for $w \in \partial E$, $g(\phi(w)) = 0$. Moreover, $w - f(\phi(w)) = a \in E$ is a constant, so we can apply Lemma 5.2 to $f \circ \phi$, $g \circ \phi$ and $h \circ \phi$ to obtain $w \in E$ such that $(f + gh)(\phi(w)) = 0$.

Note: it suffices that $h \in C(\phi[E]; \mathbb{C}^n)$. □

We conclude that all of our noncausal subalgebras of $C(\mathbb{T}^m)$ have bsr $> m/2$.

Corollary 5.7. *Let $\mathcal{A} \subset C(\mathbb{T}^m)$ be a function algebra. If $z_j, z_j^{-1} \in \mathcal{A}$ $(j = 1, 2, \ldots, m)$, then* $\operatorname{bsr} \mathcal{A} \geq \lfloor m/2 \rfloor + 1$.

Note that here $\mathcal{A}$ is either a real algebra or a subalgebra of $C(\mathbb{T}^m)$.

Proof. This follows directly from Corollary 5.6 if $m = 2n$. Assume then that $m = 2n + 1$. Set $f'(z, w) := f(z)$, $g'(z, w) := g(z)$ $(z \in \mathbb{T}^{2n},\ w \in \mathbb{T})$. Then $f'_j, g' \in \mathcal{A}$ $(j \leq n)$. Given $h' \in \mathcal{A}^m$, set $h(z) := h'(z, 1)$ $(z \in \mathbb{T}^{2n})$ to have $0 = (f + hg)(z)$ for some $z \in \mathbb{T}_r^{2n}$ and so $0 = (f' + h'g')(z, 1)$. Thus $f' + h'g' \notin \mathrm{U}_n(\mathcal{A})$. □

A converse is given in Theorem 5.13.

To construct our ultimate nonreducible unimodular vector, we need the following technical lemma. It uses the fact that $\phi(z) := \tan \frac{\pi}{4} z$ maps $[-1, 1] \to [-1, 1]$ but shrinks the vertical direction.

Lemma 5.8. *Set $\phi(z) := \tan \frac{\pi}{4} z$, $\psi(z) := i\phi(-iz)$. Then ϕ and ψ are holomorphic on a neighborhood of $\overline{\mathbb{D}}$, $\phi(z) \in \mathbb{D}$ when $-1 < \operatorname{Re} z < 1$, $\psi(z) \in \mathbb{D}$ when $-1 < \operatorname{Im} z < 1$, $\phi(\pm 1) = \pm 1$, and $\psi(\pm i) = \pm i$. In particular, $\operatorname{Re} \psi(z) \in (-1, 1)$ and $\operatorname{Im} \phi(z) \in (-1, 1)$ when $z \in \overline{\mathbb{D}}$. Moreover, $\phi(\overline{z}) = \overline{\phi(z)}$, $\psi(\overline{z}) = \overline{\psi(z)}$.*

Finally, there exists $\epsilon > 0$ such that for every $r \in (0, \epsilon)$ we have

$$\begin{aligned} &x + \operatorname{Re} r\phi(\mathrm{e}^{ix}) \in [1, \pi - 1] \ (x \in [0, \pi]), \\ &x + \pi/2 - \operatorname{Im} r\psi(\mathrm{e}^{ix}) \in [1, \pi - 1] \ (x \in [-\pi/2, \pi/2]). \end{aligned} \tag{8}$$

Proof. Most claims are straightforward (note that ϕ has poles at $2 + 4n$ $(n \in \mathbb{Z})$ and that sin and cos are real-symmetric), and so we only prove (8). Since $\phi(-z) = -\phi(z)$ $(z \in \overline{\mathbb{D}})$, we have $\operatorname{Im} r\psi(\mathrm{e}^{i(x-\pi/2)}) = \operatorname{Im} r\psi(-i\mathrm{e}^{ix}) = \operatorname{Im} ri\phi(-\mathrm{e}^{ix}) = -\operatorname{Re} r\phi(-\mathrm{e}^{ix})$, and so the latter claim in (8) follows from the former, which we establish below.

The function $g(x) := x + r\phi(\mathrm{e}^{ix})$ has $g'(x) = 1 + ir\mathrm{e}^{ix}\phi'(\mathrm{e}^{ix})$. Thus we have $\operatorname{Re} g'(x) \geq 0$ $(x \in [0, \pi])$ once we require that r is sufficiently small (which we do). But $\operatorname{Re} g(0) = 1 + r$, so $g(x) \geq 1$ $(x \in [0, \pi])$. Now $\operatorname{Re} g(x) \leq \pi/2 + r \leq \pi - 1$ $(x \in [0, \pi/2])$ if r is small. Hence $\operatorname{Re} g(x) \in [1, \pi - 1]$ $(x \in [0, \pi/2])$.

Given $x \in [0, \pi/2]$, set $z := \mathrm{e}^{ix}$, $w := \mathrm{e}^{i(\pi - x)} = -\overline{z}$. Since $\phi(-\overline{z}) = -\overline{\phi(z)}$, we have $\operatorname{Re} \phi(w) = -\operatorname{Re} \phi(z)$, and so

$$g(\pi - x) = \pi - x + r \operatorname{Re} \phi(\mathrm{e}^{i(\pi - x)}) = \pi - (x + r \operatorname{Re} \phi(\mathrm{e}^{ix})) = \pi - \operatorname{Re} g(x) \in [1, \pi - 1].$$

Consequently, $x + \operatorname{Re} r\phi(\mathrm{e}^{ix}) \in [1, \pi - 1]$ $\forall x \in [0, \pi]$. □

Now we can construct a unimodular vector $(f, g) \in \mathrm{U}_{k+1}(\mathcal{F}\ell^1(\mathbb{N}^m; \mathbb{R}))$ that is not "reducible on $\mathbb{T}^m$", hence nor in many algebras, as noted later below. So we shall have f, g holomorphic on $\mathbb{D}^m$ (unlike in Corollary 5.6) and it suffices that h is defined on (a part of) $\mathbb{T}^m$ (unlike in Corollary 5.3). The price that we pay is the fact that f is not a polynomial.

Lemma 5.9. *Let $0 \neq k \in \mathbb{N}$, $m := 2k$. Define $g, f_j \in C(\overline{\mathbb{D}}^m; \mathbb{C})$ by*

$$g(z) := \prod_{j=1}^{k} (z_{2j-1}^2 - 1)(z_{2j}^2 + 1) \qquad (z \in \overline{\mathbb{D}}^m), \tag{9}$$
$$f_j(z) := -r\phi(z_{2j-1}) + r\psi(z_{2j}) \qquad (j = 1, \ldots, k,\ z \in \overline{\mathbb{D}}^m),$$

where ϕ, ψ and $r \leq 1$ are as in Lemma 5.8. Set $E_1 := [0, \pi] \times [-\pi/2, \pi/2]$, $E := E_1^k \subset \mathbb{R}^m$, $E_m := \mathrm{e}^{iE} \subset \mathbb{T}^m$. If $h \in C(E_m; \mathbb{C}^k)$, then $(f + gh)(z) = 0$ for some $z \in E_m$. Moreover, $(f, g) : \overline{\mathbb{D}}^m \to \mathbb{C}^{k+1}$ has no zeros on $\overline{\mathbb{D}}^m$, and f and g are absolutely converging sums of real-coefficient polynomials on a neighborhood of $\overline{\mathbb{D}}^m$, hence $(f, g) \in \mathrm{U}_{k+1}(\mathcal{F}\ell^1(\mathbb{N}^m; \mathbb{R}))$ is not reducible.

Proof.

1° Let $z \in \overline{\mathbb{D}}^m$. We have $g(z) = 0$ iff $z_{2j} = \pm i$ or $z_{2j-1} = \pm 1$ for some $j \leq k$. If $z_{2j-1} = \pm 1$, then $\operatorname{Re} f_j(z)/r = \mp 1 + \operatorname{Re} \psi(z_{2j}) \in \mp 1 + (-1, 1) = \mp(0, 2) \not\ni 0$, hence then $f_j(z) \neq 0$. Similarly, if $z_{2j} = \pm i$, then $\operatorname{Im} f_j/r \in (-1, 1) \pm 1 \not\ni 0$. Consequently, $g(z) = 0 \Rightarrow f(z) \neq 0$ for every $z \in \overline{\mathbb{D}}^m$. Moreover, g is a polynomial and ϕ and ψ are holomorphic (hence their MacLaurin polynomials converge absolutely) on a neighborhood of $\overline{\mathbb{D}}$, by Lemma 5.8, which also shows that ϕ and ψ are real-symmetric (that is, they have real coefficients), so we have proved the last sentence (use Theorem 4.5 for unimodularity).

2° Set $G_f(x) := f(\mathrm{e}^{ix})$, $G_g(x) := g(\mathrm{e}^{ix})$, $G_h(x) := h(\mathrm{e}^{ix})$, $(x \in E)$, where $\mathrm{e}^{ix} := (\mathrm{e}^{ix_1}, \ldots, \mathrm{e}^{ix_m})$. Then $G_f, G_h \in C(E; \mathbb{C}^k)$, $G_g \in C(E; \mathbb{C})$. We shall soon apply Lemma 5.2 to G_f, G_g and G_h. From 1° we observe that $G_g = 0$ on ∂E.

3° We have $x - G_f(x) \in E$ for every $x \in E$. Indeed, let $x \in E$. Set $w_j := x_{2j-1} + ix_{2j}$ ($j \leq k$), so that $w \in \mathbb{C}^k$ is identified with $x \in \mathbb{R}^m$. Set $z := \mathrm{e}^{ix}$. Then $z_{2j-1} = \mathrm{e}^{ix_{2j-1}}$, and so $\operatorname{Re} w_j - \operatorname{Re} G_f(w)_j = x_{2j-1} - \operatorname{Re} f_j(z) = (x_{2j-1} + \operatorname{Re} r\phi(z_{2j-1})) - r \operatorname{Re} \psi(z_{2j}) \in [1, \pi - 1] + r[-1, 1] \subset [0, \pi]$, by (8). Similarly, $\operatorname{Im} w_j - \operatorname{Im} G_f(w)_j = x_{2j} - \operatorname{Im} r\psi(z_{2j}) + r \operatorname{Im} \phi(z_{2j-1}) \in [1, \pi - 1] - \pi/2 + r[-1, 1] \subset [-\pi/2, \pi/2]$. Thus $w_j - G_f(w)_j \in E_1$. Since this holds for any $j \leq k$, we have $w - G_f(w) \in E$.

4° By 2° and 3°, we have $F \in C(\partial E; E)$, where $F(x) := x - G_f(x)$. Identify $E \subset \mathbb{R}^m$ with $E \subset \mathbb{C}^k$ (and consider G_f, G_g and G_g as defined on the latter). By Lemma 5.2 (and Lemma 5.1), there exists $x \in E$ such that $(G_f + G_g G_h)(x) = 0$. Set $z := \mathrm{e}^{ix} \in \mathbb{T}^m$ to complete the proof.

□

Corollary 5.10. *Let $\mathcal{A} \subset C(\mathbb{T}^n)$ be a function algebra. If $\mathcal{F}\ell^1(\mathbb{N}^n; \mathbb{R}) \subset \mathcal{A}$, then* $\operatorname{bsr} \mathcal{A} \geq \lfloor n/2 \rfloor + 1$.

Proof. The proof of Corollary 5.7 applies here too, mutatis mutandis, with Lemma 5.9 in place of Corollary 5.6. Use also the fact that $(f, g) \in \mathrm{U}_{k+1}(\mathcal{F}\ell^1(\mathbb{N}^n; \mathbb{R}))$ by the corona theorem 4.5, and observe that $\mathrm{U}_{k+1}(\mathcal{F}\ell^1(\mathbb{N}^n; \mathbb{R})) \subset \mathrm{U}_{k+1}(\mathcal{A})$. □

The assumptions of Corollary 5.10 are weakened in Lemma 9.7, a converse is given in Theorem 5.13, and a related result is given in Corollary 6.3.

5.2. **bsr and tsr.** Practically all topological algebras of holomorphic functions (except $\mathbb{K}1$) have $\operatorname{tsr} \geq 2$, as the proof below shows.

Lemma 5.11. *Let $\mathcal{A} \subset H^\infty(\Omega)$ be a function algebra with topology. Assume that the sup-norm is continuous on $\mathcal{A}$. If some $f \in \mathcal{A} \neq \{0\}$ has a zero on Ω, then* $\operatorname{tsr} \mathcal{A} \geq 2$.

Proof. If $f(z) = 0$ and $f_n \to f$ in $\mathcal{A}$, as $n \to \infty$, then the function f_n has a zero near z for each big n, by Hurwitz' Theorem. Therefore, $f \notin \overline{\mathrm{U}_1(\mathcal{A})}$, so $\operatorname{tsr} \mathcal{A} > 1$. □

From [8] we conclude the following.

Lemma 5.12. $\operatorname{bsr} \ell^1(\mathbb{N}^n) = \lfloor n/2 \rfloor + 1$.

Proof. Let $\mathcal{A} = \ell^1(\mathbb{N}^n)$. By Corollary 5.10 (or [8, Theorem 3.12] or [28, Theorem 3.3]), $\operatorname{bsr} \mathcal{F}\mathcal{A} \geq \lfloor n/2 \rfloor + 1$. By [8, p. 543], $\operatorname{bsr} \mathcal{A} \leq \operatorname{dsr} \mathcal{A}$, and by [8, Theorem 3.4], $\operatorname{dsr} \mathcal{A} \leq \lfloor n/2 \rfloor + 1$ (since it is a n-generated unital complex commutative Banach algebra). Here dsr stands for "dense stable rank"; the interested reader is referred to [8] for the definition and background of this concept. □

Recall that $C(\mathbb{T}^n)_\mathbb{R} = \{f \in C(\mathbb{T}^n) \colon f(\bar{z}) = \overline{f(z)}\ \forall z \in \mathbb{T}^n\}$ is a real Banach algebra. Set $1/z := (z_1^{-1}, \dots, z_n^{-1})$ for all $z \in (\mathbb{C} \setminus \{0\})^n$. Now we show that its typical noncausal subalgebras have bsr = tsr = $\lfloor n/2 \rfloor + 1$.

Theorem 5.13. *Let $\mathcal{A} \subset C(\mathbb{T}^n)$ be a topological function algebra containing all real polynomials. Assume that some functions holomorphic on a neighborhood $\mathbb{T}^n$ are dense in $\mathcal{A}$. Assume that $f(1/\cdot) \in \mathcal{A}$ for all $f \in \mathcal{A}$. Assume also that for any $k \geq 1$ a function $f \in \mathcal{A}^k$ is unimodular if $f(z) \neq 0$ for all $z \in \mathbb{T}^n$. Then* $\operatorname{bsr} \mathcal{A} = \operatorname{tsr} \mathcal{A} = \lfloor n/2 \rfloor + 1$.

Proof. By Corollary 5.7 and Proposition 3.2 we have $\lfloor n/2 \rfloor + 1 \leq \operatorname{bsr} \mathcal{A} \leq \operatorname{tsr} \mathcal{A}$. So it remains to show that $\operatorname{tsr} \mathcal{A} \leq \lfloor n/2 \rfloor + 1$, and we do this below.

(The proof below could be simplified if we assumed that $\mathcal{A}$ is a complex (not real) algebra. The same applies to Theorem 5.16. The reader may wish to first read the proof of Lemma 8.5, which is a simplified version of this.)

Set $k := \lfloor n/2 \rfloor + 1$. Then $2k \geq n + 1$. Let $f \in \mathcal{A}^k$ be holomorphic on a neighborhood of $\mathbb{T}^n$. We shall show that $f \in \overline{\mathrm{U}_k}$ (it follows that $\operatorname{tsr} \mathcal{A} \leq k$). Set

$$\mathcal{P} := \{p \in \{-1, 0, 1\}^n \colon p_j = 0\ \forall j > k\}.$$

For each $p \in \mathcal{P}$ and $j \in \{1, \dots, n\}$ we define

$$E_j^p := \begin{cases} \{p_j\}, & \text{if } p_j \neq 0; \\ \mathbb{T} \setminus \{\pm 1\}, & \text{if } p_j = 0,\ j \leq k; \\ \mathbb{T}, & \text{if } j > k; \end{cases} \qquad g_j^p(z) := \begin{cases} 0, & \text{if } p_j \neq 0 \text{ or } j > k; \\ \operatorname{Im} f_j(z) / \operatorname{Im} z_j, & \text{if } p_j = 0 \text{ and } j \leq k. \end{cases}$$

Obviously, $\mathbb{T}^n = \cup_{p \in \mathcal{P}} E^p$, where $E^p := \prod_{j=1}^n E_j^p$. By $\#p$ we denote the number of j such that $p_j \neq 0$. Set $F_p := (\operatorname{Re} f; g^p) \in C^1(E^p; \mathbb{R}^{2k})$ $(p \in \mathcal{P})$ (with slight abuse of notation, similarly below). Since E^p is $n - \#p$-dimensional, by Lemma A.2 we have[3] $m_{n+1-\#p}(F_p[E^p]) = 0$, and so $m_{n+1}(F_p[E^p] \times \mathbb{R}^{\#p}) = 0$. From the above $m_{n+1}(N_p) = 0$, where

$$N_p := \left\{ \{\operatorname{Re} f(z)\} \times \prod_{j=1}^k \begin{cases} \mathbb{R}, & \text{if } p_j \neq 0; \\ \{g_j^p(z)\}, & \text{if } p_j = 0 \end{cases} \quad : \quad z \in E^p \right\} \subset \mathbb{R}^{2k}.$$

---

[3]Here, by $m_j(A) = 0$, we mean that A is the C^1 image of some $A' \subset \mathbb{R}^j$ satisfying $m_j(A') = 0$. By Lemma A.2, this is equivalent to the standard definition if $A \subset \mathbb{R}^j$.

Thus, $m_{2k}(N_p) \le m_{n+1}(N_p) = 0$ (for any $p \in \mathcal{P}$). Set $N := \cup_{p\in\mathcal{P}} N_p \subset \mathbb{R}^{2k}$. Then $m_{2k}(N) = 0$, and so we can fix an arbitrarily small (by absolute value) $(r;t) \in \mathbb{R}^{2k} \setminus N$. Define $h \in C^\infty(\mathbb{T}^n;\mathbb{R}^{2k})$ by

$$h(z) := (\operatorname{Re} f(z); \operatorname{Im} f(z)) - (r; t_1 \operatorname{Im} z_1, \dots, t_k \operatorname{Im} z_k).$$

We now assume that $h(z) = 0$ for some $z \in \mathbb{T}^n$ and derive a contradiction. Pick $p \in \mathcal{P}$ such that $z \in E^p$. Then $\operatorname{Re} f(z) = r$ and $g_j^p(z) = t_j$ for those $j \le k$ for which $p_j = 0$. Consequently, $(r,t) \in N_p$, a contradiction. Thus h has no zeros.

Hence the function

$$G := f - r - i(t_1 \operatorname{Im} z_1, \dots, t_k \operatorname{Im} z_k) : \mathbb{T}^n \to \mathbb{C}^k$$

has no zeros (because $h = (\operatorname{Re} G, \operatorname{Im} G)$). But $G \in \mathcal{A}^k$, since $i \operatorname{Im} z_j = (z_j - z_j^{-1})/2 \in \mathcal{A}$, and so $G \in \mathrm{U}_k(\mathcal{A})$ (having no zeros). As G was arbitrarily close to f, we have $\operatorname{tsr} \mathcal{A} \le k$. □

The above applies to all of our "noncausal n-dimensional classes".

Corollary 5.14. *We have* $\operatorname{bsr} \mathcal{A} = \operatorname{tsr} \mathcal{A} = \lfloor n/2 \rfloor + 1$ *when* $\mathcal{A}$ *equals any of* $C(\mathbb{T}^n)$, $C(\mathbb{T}^n)_{\mathbb{R}}$, $\ell^1(\mathbb{Z}^n)$, $\ell^1(\mathbb{Z}^n;\mathbb{R})$, $\mathcal{M}^{(n)}$ *and* $(\mathcal{M}^{(n)})_{\mathbb{R}}$.

Proof. By Remark 2.1, $\mathcal{M}^{(n)}$ and $(\mathcal{M}^{(n)})_{\mathbb{R}}$ may be omitted, so the corollary follows from Theorem 5.13. Indeed, the real or complex polynomials in $z_1, \dots, z_n; z_1^{-1} \dots, z_n^{-1}$ are dense $C(\mathbb{T}^n)$, $C(\mathbb{T}^n)_{\mathbb{R}}$ and $\mathcal{F}\ell^1$'s, by Lemma A.1; by the Corona Theorems 4.3 and 4.5, also the unimodularity condition is satisfied; obviously also the other assumptions of Theorem 5.13 are satisfied. □

Corollary 5.15. *Let* $\mathcal{A} \subset C(\mathbb{T}^n)$ *be a full subalgebra. Assume that* $z_k, z_k^{-1} \in \mathcal{A}$ *for* $k = 1, 2, \dots, n$. *Then* $\operatorname{bsr} \mathcal{A} = \lfloor n/2 \rfloor + 1$.

Note that $\mathcal{A}$ being full in $C(\mathbb{T}^n)$ means that $f^{-1} \in \mathcal{A}$ if $f \in \mathcal{A}$ and $f(z) \ne 0$ for all $z \in \mathbb{T}^n$.

Proof. Assume first that $\mathcal{A}$ is complex. Since $\mathcal{A}$ is dense in $C(\mathbb{T}^n)$ (Lemma A.1), from Lemma 3.6, it follows that $\operatorname{bsr} \mathcal{A} \le \operatorname{bsr} C(\mathbb{T}^n) = \lfloor n/2 \rfloor + 1$. But $\operatorname{bsr} \mathcal{A} \ge \lfloor n/2 \rfloor + 1$, by 2° of the proof of Theorem 5.13. The real case is analogous (replace $C(\mathbb{T}^n)$ by $C(\mathbb{T}^n)_{\mathbb{R}}$). □

Next we study the "causal case". Recall that by $A(\mathbb{D}^n)$ we denote the *polydisc algebra* of continuous functions $\overline{\mathbb{D}}^n \to \mathbb{C}$ that are holomorphic on $\mathbb{D}^n$. For this algebra and its typical subalgebras we have $\operatorname{tsr} \mathcal{A} = n + 1$; in the real case also $\operatorname{bsr} \mathcal{A} = n + 1$.

Theorem 5.16. *Let* $K \subset \mathbb{C}^n$ *and suppose that* $\mathcal{A} \subset C(K)$ *is a topological function algebra containing all real polynomials. Assume that some functions holomorphic on a neighborhood* K *are dense in* $\mathcal{A}$. *Assume also that a function* $f \in \mathcal{A}^{n+1}$ *is unimodular if* $f(z) \ne 0$ *for all* $z \in K$. *Then* $\operatorname{tsr} \mathcal{A} \le n + 1$. *If* K *has a nonempty interior and the sup norm is continuous on* $\mathcal{A}$, *then* $\operatorname{tsr} \mathcal{A} = n + 1$. *If* $\mathcal{A} \subset C(K)_{\mathbb{R}}$, *and* $\bar{B}_r(a) := \{x \in \mathbb{R}^n : |x - a| \le r\} \subset K$ *for some* $a \in \mathbb{R}^n$ *and* $r > 0$, *then* $\operatorname{bsr} \mathcal{A} = \operatorname{tsr} \mathcal{A} = n + 1$.

In the last claim we have set $C(K)_{\mathbb{R}} := \{f \in C(K) \colon f(x) \in \mathbb{R}\ \forall x \in K \cap \mathbb{R}^n\}$. One can verify that $A(\mathbb{D}^n)_{\mathbb{R}} = C(\overline{\mathbb{D}}^n)_{\mathbb{R}}$.

Proof.

1° (This is similar to the proof of Theorem 5.13.) Let $f \in \mathcal{A}^{n+1}$ be holomorphic on a neighborhood V of K. Let P be the collection of sets $p \subset \{1, \dots, n+1\}$ such that either $\{1, n+1\} \subset p$ or $\{1, n+1\} \subset P \setminus p$ (this way we get no problems from defining $z_{n+1} := z_1$ below). Set $z_{n+1} := z_1$ for any $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. For any $p \in P$ define $Q^p := \prod_{j=1}^n Q_j^p$,

$$Q_j^p := \begin{cases} \mathbb{R}, & \text{if } j \in p; \\ \{0\}, & \text{if } j \notin p, \end{cases} \quad g_j^p(z, q) := \begin{cases} q_j, & \text{if } j \in p; \\ \operatorname{Im} f_j(z)/\operatorname{Im} z_j, & \text{if } j \notin p, \end{cases}$$

and $V^p := \{z \in V \colon \operatorname{Im} z_j = 0 \ \forall j \in p \text{ and } \operatorname{Im} z_j \neq 0 \ \forall j \notin p\}$. Obviously, $V = \cup_{p \in P} V^p$. By $\#p$ we denote the number of elements in p. Set for $p \in P$, $j \in P$,

$$\begin{aligned} f_j^p(z, q) &:= \operatorname{Re} f_j(z) - g_j^p(z, q) \operatorname{Re} z_j, \\ F_p &:= (f^p; g^p) \in C(V^p \times Q^p; \mathbb{R}^{2n+2}). \end{aligned}$$

Since V^p (respectively, Q^p) is diffeomorphic to an open subset of $\mathbb{R}^{2n-\#p}$ (respectively, $\mathbb{R}^{\#p}$), we observe that F_p has the same range as a C^1 function on an open subset of $\mathbb{R}^{2n-\#p} \times \mathbb{R}^{\#p} = \mathbb{R}^{2n}$, so $m_{2n+2}(N_p) = 0$, where $N_p := F_p[V^p \times Q^p]$, by Lemma A.2. Set $N := \cup_{p \in P} N_p$. Then $m_{2n+2}(N) = 0$, so we can fix an arbitrarily small (by absolute value) $(r; t) \in \mathbb{R}^{2(n+1)} \setminus N$. Set

$$G(z) := f(z) - r - (t_1 z_1, \dots, t_{n+1} z_{n+1}) \quad (z \in K)$$

(recall that $z_{n+1} := z_1$). Then $G \in \mathcal{A}^{n+1}$ is arbitrarily close to f.

We now assume that $G(z) = 0$ for some $z \in V$ and derive a contradiction. Pick $p \in P$ such that $z \in V^p$. Define $q \in Q^p$ by $q_j := t_j$ if $j \in p$ (that is, if $\operatorname{Im} z_j = 0$), and by $q_j := 0$ otherwise. Then, for any $j = 1, \dots, n+1$, we have $0 = \operatorname{Im} G_j(z) = \operatorname{Im} f_j(z) - t_j \operatorname{Im} z_j$, hence $g_j^p(z, q) = t_j$, and $0 = \operatorname{Re} G_j(z) = \operatorname{Re} f_j(z) - r_j - t_j \operatorname{Re} z_j = f_j^p(z, q) - r_j$, hence $f_j^p(z, q) = r_j$. Therefore, $(r, t) = F_p(z, q) \in N_p$, a contradiction. Hence $G(z) \neq 0$ for all $z \in V$. Consequently, $G \in \mathrm{U}_{n+1}(\mathcal{A})$. Since G was arbitrarily close to f, $\operatorname{tsr} \mathcal{A} \leq n+1$.

2° Assume that $K^\circ \neq \emptyset$. Pick $a \in K^\circ$. Set $g_j(z) := z_j - a$ $(j \leq n)$. Then $g \in \mathcal{A}^n$. Given $f \in \mathcal{A}^n$, set $F_f(r, t) := (\operatorname{Re} f(r + it), \operatorname{Im} f(r + it))$. Then $F_g \in C^\infty(V; \mathbb{R}^{2n})$, where $V := \{(\operatorname{Re} z, \operatorname{Im} z) \colon z \in K^\circ\} \subset \mathbb{R}^{2n}$ is a neighborhood of $a_0 := (\operatorname{Re} a, \operatorname{Im} a)$. Since $F_g(a_0) = 0$ and $F_g'(a_0) = I$, the function $F_g + h$ has a zero when $\|h\|_\infty < \delta$, where δ is from Lemma A.5. If $f \in \mathcal{A}^n$ is sufficiently close to 0, then $\|f\|_\infty < \delta$ and so $\|F_f\|_\infty < \delta$. It follows that $F_{g+f} = F_g + F_f$ has a zero $(r, t) \in V$, so $r + it \in K^\circ$ and $(g + f)(r + it) = 0$. Hence $g + f \notin \mathrm{U}_n(\mathcal{A})$. Thus $g \notin \overline{\mathrm{U}_n}$, and so $\operatorname{tsr} \mathcal{A} > n$.

3° Assume now that $\mathcal{A} \subset C(K)_\mathbb{R}$ and $E := B_r(a) \subset K$. In Corollary 5.3 we have $(f, g) \in \mathrm{U}_n(\mathcal{A})$. If $h \in \mathcal{A}^n \subset C(K)_\mathbb{R}^n$, then $h|_E \in C(E; \mathbb{R}^n)$. Hence $f + hg$ has a zero, by Corollary 5.3. Thus (f, g) is not reducible. So $\operatorname{bsr} \mathcal{A} \geq n+1$. By Proposition 3.7 and 1°, we get that $n + 1 \leq \operatorname{bsr} \mathcal{A} \leq \operatorname{tsr} \mathcal{A} \leq n+1$.

□

For bsr, Theorem 5.13 only gives an upper bound in the complex case; a lower bound is given in Corollary 5.10 (which can be adapted to most cases). Exact results for some complex cases were given in [8] using the Arens–Taylor–Novodvorski theory; we present further results in Corollary 5.18 below.

Corollary 5.17. *We have* $\operatorname{tsr}\mathcal{A} = n+1$ *and* $\operatorname{bsr}\mathcal{A} = \lfloor n/2 \rfloor + 1$ *when* $\mathcal{A}$ *equals any of* $A(\mathbb{D}^n)$, $\ell^1(\mathbb{N}^n)$, *and* $\mathcal{M}_+^{(n^+)}$. *Moreover,* $\operatorname{bsr}\mathcal{A} = \operatorname{tsr}\mathcal{A} = n+1$ *when* $\mathcal{A}$ *equals any of* $A(\mathbb{D}^n)_\mathbb{R}$, $\ell^1(\mathbb{N}^n;\mathbb{R})$, *and* $(\mathcal{M}_+^{(n^+)})_\mathbb{R}$.

Proof. We may omit the proofs for $\mathcal{M}_+^{(n^+)}$ and $(\mathcal{M}_+^{(n^+)})_\mathbb{R}$, by Remark 2.1. We have $\operatorname{bsr}\ell^1(\mathbb{N}^n) = \lfloor n/2 \rfloor + 1$, by Lemma 5.12, and $\operatorname{bsr} A(\mathbb{D}^n) = \lfloor n/2 \rfloor + 1$, by [8, Corollary 3.13]. Polynomials (hence also $\mathcal{F}\ell^1$) are dense in $\mathcal{A}$ or in $\mathcal{F}\mathcal{A}$, and the Corona Theorems 4.3 and 4.5 provide the remaining assumptions of Theorem 5.16, which yield the remaining claims. □

Below fullness obviously means that if $f \in \mathcal{A}$ and $f \neq 0$ on $\overline{\mathbb{D}}^n$, then $f^{-1} \in \mathcal{A}$, that is, that $\mathcal{A}$ is "inversionally closed".

Corollary 5.18. *If* $\mathcal{A}$ *is a full subalgebra of* $A(\mathbb{D}^n)$ *(respectively, of* $A(\mathbb{D}^n)_\mathbb{R}$*), and* $\mathcal{A}$ *contains the polynomials, then* $\operatorname{bsr}\mathcal{A} = \lfloor n/2 \rfloor + 1$ *(respectively,* $\operatorname{bsr}\mathcal{A} = n+1$*).*

Proof. By Lemma 3.6 (and Lemma A.1 and Corollary 5.17), we have $\operatorname{bsr}\mathcal{A} \le \operatorname{bsr} A(\mathbb{D}^n) = \lfloor n/2 \rfloor + 1$ (respectively, $\operatorname{bsr}\mathcal{A} \le \operatorname{bsr} A(\mathbb{D}^n)_\mathbb{R} = n+1$). By [28, Theorem 3.3] (respectively, Corollary 5.4), we have $\operatorname{bsr}\mathcal{A} \ge \lfloor n/2 \rfloor + 1$ (respectively, $\operatorname{bsr}\mathcal{A} \ge n+1$).

Actually, [28] treats the unit ball, and there is a slight mistake in its proof, so we rewrite Rupp's proof here for $\mathbb{D}^n$. Set $B_n := \{z \in \mathbb{C}^n \colon |z| \le 1\} \subset \overline{\mathbb{D}}^n$. Then $A(\mathbb{D}^n) \subset A(B_n)$, where $A(B_n)$ stands for continuous functions $B_n \to \mathbb{C}$ that are holomorphic in the interior of B_n.

Set $k := \lfloor n/2 \rfloor$. Define $f = (f_1, f_2, \ldots, f_k) \in A(\mathbb{D}^n)^k$ by $f_j(z) := z_j$. Set

$$g(z) := z_1 z_{k+1} + \ldots + z_k z_{2k} - 1/4.$$

Then $z_{k+1} f_1 + \ldots + z_{2k} f_k - g = 1/4 \in \mathrm{U}_1$, so $(f, g) \in \mathrm{U}_{k+1}(\mathcal{A})$. If $h : B_n \to \mathbb{C}^k$ is continuous, then $(f + hg)(\zeta) = 0$ for some $\zeta \in \overline{\mathbb{D}}^n$. Indeed, define $f_j, h_j, g \in C(B_k;\mathbb{C})$ by

$$f'(z) := f(z, \bar{z}, 0), \quad g'(z) := g(z, \bar{z}, 0) = |z|^2 - 1/4, \quad h'(z) := h(z, \bar{z}, 0)$$

(remove the 0's if $n = 2k$) to obtain, from Lemma 5.2, a $z \in E := B_k/2$ such that $(f + hg)(z, \bar{z}, 0) = (f' + h'g')(z) = 0$, so that (f, g) is not reducible (since $(z, \bar{z}, 0) \in B_n \subset \overline{\mathbb{D}}^n$). □

Corollaries 5.17 and 5.18 also hold with the open unit ball $\{z \in \mathbb{C}^n \colon |z| < 1\}$ in place of $\mathbb{D}^n$, with the same proofs; this sharpens [28, Theorem 3.3] (using partially Rupp's result).

Next we compute the bsr's and tsr's for the $\mathbb{K} + \mathrm{L}^1$ classes, using the above theorems.

From [5, pp. 62–63] one gets the following lemma; in particular, that real-coefficient rational functions are dense in $\mathcal{F}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R};\mathbb{R}))$.

Lemma 5.19. *[Real] linear combinations of* $(\cdot + 1)^{-k}$ *(respectively, at* $(\cdot \pm 1)^{-k}$*)* $(k = 0, 1, 2, \ldots)$ *form a dense subset of [real-symmetric elements of]* $\mathcal{F}(\mathbb{C}\delta_0 + \mathrm{L}^1_+)$ *(respectively, of* $\mathcal{F}(\mathbb{C}\delta_0 + \mathrm{L}^1)$*).*

(The alternative real claim follows from the complex one by taking real parts in $\mathbb{C}\delta_0 + \mathrm{L}^1_+$ (respectively, in $\mathbb{C}\delta_0 + \mathrm{L}^1$).)

Because the Cayley transform maps $-1 \mapsto \infty$ and $1 \mapsto 0$, Lemma 5.19 and the corona theorems lead to the following.

Lemma 5.20. *Let* $\mathcal{A}$ *denote* $\mathbb{C}\delta_0 + \mathrm{L}^1_+$ *(respectively,* $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$, $\mathbb{C}\delta_0 + \mathrm{L}^1$, $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R};\mathbb{R})$*) and let* $\mathcal{A}'$ *denote* $A(\mathbb{D})$ *(respectively,* $A(\mathbb{D})_\mathbb{R}$, $C(\mathbb{T})$, $C(\mathbb{T})_\mathbb{R}$*). Then the embedding*

$$f : \mathcal{A} \ni g \mapsto \widehat{g} \circ \phi \in \mathcal{A}' \tag{10}$$

where ϕ is the Cayley transform $z \mapsto (1-z)/(1+z)$, *is a continuous algebra homomorphism, and the polynomials in z (respectively, real polynomials in z, polynomials in z and z^{-1}, real polynomials in z and z^{-1}) form a dense subset of $f[\mathcal{A}]$ (in the topology coinduced from $\mathcal{A}$). An element of $f[\mathcal{A}]^n$ is unimodular iff it has no zeros on $\overline{\mathbb{D}}$ (respectively, $\overline{\mathbb{D}}$, $\mathbb{T}$, $\mathbb{T}$). In particular,* (10) *satisfies the assumptions of Corollary 3.8.*

For $\mathcal{A}$ equal to $\mathbb{C}\delta_0 + \mathrm{L}^1$ or $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R};\mathbb{R})$, we also have $h(1/\cdot) \in f[\mathcal{A}]$ for every $h \in f[\mathcal{A}]$.

Proof. The containment and density of polynomials was essentially explained below Lemma 5.19. The sup-norm is continuous on $\mathcal{A}$ and invariant under f, and so f is continuous. The unimodularity claim follows from the corona Theorems 4.3 and 4.5. The last claim follows from the facts that $f(g)(1/z) = f(g(-\cdot))(z)$ and $g(-\cdot) \in \mathcal{A}$ if $g \in \mathcal{A}$, $z \in \mathbb{T}$. □

Theorem 5.13 applies to $\mathbb{C}\delta_0 + \mathrm{L}^1$ and to $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R};\mathbb{R})$ as well, by Lemma 5.20.

Corollary 5.21. *We have* $\mathrm{bsr}(\mathbb{C}\delta_0 + \mathrm{L}^1) = \mathrm{tsr}(\mathbb{C}\delta_0 + \mathrm{L}^1) = 1$, *and* $\mathrm{bsr}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R};\mathbb{R})) = \mathrm{tsr}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R};\mathbb{R})) = 1$.

From Lemma 5.20 and [29, Remark, p. 87] we conclude that $\mathrm{bsr}(\mathbb{C}\delta_0 + \mathrm{L}^1_+) = 1$. By Lemma 5.20, Theorem 5.16 (with $K := \overline{\mathbb{D}}$) applies to $\mathbb{C}\delta_0 + \mathrm{L}^1_+$ and to $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$ as well. This implies the following.

Corollary 5.22. *We have* $\mathrm{bsr}(\mathbb{C}\delta_0 + \mathrm{L}^1_+) = 1$, $\mathrm{tsr}(\mathbb{C}\delta_0 + \mathrm{L}^1_+) = 2$, *and* $\mathrm{bsr}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})) = \mathrm{tsr}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})) = 2$.

Notes

We mention below some simpler (or more constructive) proofs for cases "$n = 1$".

For $C(\mathbb{T})$, $\ell^1(\mathbb{Z})$, $\mathcal{M}^{(1)}$ and $\mathbb{C}\delta_0 + \mathrm{L}^1$ (see Corollary 5.14), to show that tsr $= 1$, we could alternatively add a small constant $c \in \mathbb{C}$ such that $-c$ is in the range of the function/transform (use Lemma A.2 and the density of holomorphic functions). This also leads to a constructive proof of bsr $= 1$; see the end of §10.

The fact that tsr ≤ 2 for these algebras and their real counterparts can also be shown as follows: given (f, g), assume that f or $\widehat{f}$ (depending on the algebra) is holomorphic on a neighborhood of the domain (so that it has only finitely many zeros) and add a constant to g or $\widehat{g}$ so that it is nonzero at the zeros of f or $\widehat{f}$. In the causal cases we also have tsr ≥ 2, by Lemma 5.11.

As mentioned above Theorem 1.2, some of the results in this section were already known.

6. Infinitely many generators: $\mathcal{M}$, $\ell^1(\mathbb{R})$, AP, ...

In this section we prove all bsr $= \infty$ results of Table 1 and related results.

Given k, in Lemma 6.2 we construct a $k+1$-unimodular real-valued causal measure that is not reducible by any measure (nor in $\mathcal{F}^{-1}$ AP). From this we conclude in Corollaries 6.3 and 6.4 the bsr $= \infty$ results of Table 1. Naturally, other analogous results can also be concluded. Corollary 6.3 solves Mortini's problem, as mentioned in Remark 6.5.

We start from a technical lemma: the $\mathbb{Q}$-basis of the $\mathbb{Q}$-span of a finite set A can be chosen so that the coordinates of each element of A become integers.

Lemma 6.1. *If $A \subset \mathbb{R}$ is finite and the $\mathbb{Q}$-dimension $n := \dim_{\mathbb{Q}}(\mathrm{span}\, A) \geq 1$, then there are (necessarily $\mathbb{Q}$-linearly independent) $r_1, \ldots, r_n \in \mathrm{span}(A) \cap \mathbb{R}_+$ such that every $a \in A$ can be written as $\sum_{k=1}^n m_k r_k$ for some $m_1, \ldots, m_n \in \mathbb{Z}$.*

If $A \subset \mathbb{R}_+$, then it is possible to replace $\mathbb{Z}$ by $\mathbb{N}$ above; we omit the proof.

Proof. Pick a $\mathbb{Q}$-basis $r_1, \ldots, r_n$ of $\mathbb{Q}$-span$(A) := \mathbb{Q}a_1 + \ldots + \mathbb{Q}a_l$, where $A = \{a_1, a_2, \ldots, a_l\}$. Divide r_k $(k = 1, \ldots, n)$ by the product of the denominators of the coordinates m_k $(k = 1, \ldots, n)$ for each $a \in A$, and finally replace r_k by $|r_k|$. □

Now we can construct unimodular functions that are non-reducible in all of our infinite-period algebras.

Lemma 6.2. *Let* $0 \neq k \in \mathbb{N}$. *Define* $f_j, g \in \mathcal{F}\ell^1(\mathbb{N}^{2k}; \mathbb{R})$ $(j \leq k)$ *by* (9). *Let* $T_1, \ldots, T_{2k} > 0$ *be independent over* $\mathbb{Q}$. *Set* $f'(s) := f(\mathrm{e}^{-sT})$, $g'(s) := g(\mathrm{e}^{-sT})$, *where* $\mathrm{e}^{-sT} := (\mathrm{e}^{-sT_1}, \ldots, \mathrm{e}^{-sT_{2k}})$. *Then* $(f', g') \in \mathcal{F}\mathrm{U}_{k+1}((\mathcal{M}_+^{(2k^+)})_\mathbb{R})$, *but if* $h' \in \mathrm{AP}$, *then* $\inf_{i\mathbb{R}} |f' + g'h'| = 0$.

In particular, a certain coprime pair $(f', g') \in \mathcal{F}\mathrm{U}_2((\mathcal{M}_+^{(2^+)})_\mathbb{R})$ is not reducible by any $h' \in \mathrm{AP}$ (though it is reducible by some $h' \in H^\infty$, since $\operatorname{bsr} H^\infty = 1$).

Note that given independent (over $\mathbb{Q}$) "delays" $T_1, T_2 > 0$, the set $\mathcal{F}(\mathcal{M}_+^{(2^+)})_\mathbb{R}$ consists of functions $f'(s) = \sum_{j,l \in \mathbb{N}} a_{j,l} \mathrm{e}^{-s(jT_1 + lT_2)}$, where $a \in \ell^1(\mathbb{N}^2; \mathbb{R})$. From (9) we observe that in the lemma the function g is a polynomial and f is holomorphic on a neighborhood of $\overline{\mathbb{D}}^2$. Hence the above a satisfies $\sum_{j,l} r^{j+l} |a_{j,l}| < \infty$ for some $r > 1$ and g' is a finite sum of the above form (its "a" has a finite support).

Proof of Lemma 6.2: From the last sentence of Lemma 5.9 and Remark 2.1 it follows that $(f', g') \in \mathrm{U}_{k+1}((\mathcal{M}_+^{(2k^+)})_\mathbb{R})$. By density (in AP), we can assume that h' is a linear combination of a finite number of functions $\mathrm{e}^{-\cdot r'_j}$, where $r'_j \in \mathbb{R}$ $(j = 1, \ldots, n')$. Apply Lemma 6.1 to the set $S := (\cup_{j=1}^{2k} T_j) \cup (\cup_{j=1}^{n'} r'_j)$ to obtain a $\mathbb{Q}$-basis $r_1, \ldots, r_n \subset (0, \infty)$ of this set (where $n := \dim_\mathbb{Q} \operatorname{span}(S)$). Note that $A \in \mathbb{Z}^{2k \times n}$, where A is the matrix defined by the (unique) representation $T_j = \sum_{l=1}^n A_{j,l} r_l$ $(j \leq 2k)$. Set

$$T := (T_1, \ldots, T_{2k}) \in \mathbb{R}^{2k}, \quad r := (r_1, \ldots, r_n) \in \mathbb{R}^n. \tag{11}$$

Then $T = Ar$. Since each r'_j is a $\mathbb{Z}$-linear combination of r_l's (that is, $r'_j = \sum_{l=1}^n \alpha_l r_j$ for some $\alpha \in \mathbb{Z}^n$), we can write h' with the multinomial notation $r^\alpha := r_1^{\alpha_1} \cdots r_n^{\alpha_n}$ as

$$h'(s) = \sum_{\alpha \in \mathbb{Z}^n} h_\alpha \mathrm{e}^{-s \sum_{l=1}^n \alpha_l r_l} = \sum_{\alpha \in \mathbb{Z}^n} h_\alpha \prod_l (\mathrm{e}^{-sr_l})^{\alpha_l} = \sum_{\alpha \in \mathbb{Z}^n} h_\alpha (\mathrm{e}^{-sr})^\alpha \quad (s \in i\mathbb{R})$$

for some $\{h_\alpha\}_{\alpha \in \mathbb{Z}^n} \in \ell^1(\mathbb{Z}^n)$ (whose support is finite). Moreover, we have[4]

$$\beta \cdot T := \sum_{j=1}^{2k} \beta_j T_j = \beta \cdot (Ar) = r \cdot (A^\top \beta) \quad (\beta \in \mathbb{Z}^{2k}).$$

The matrix $A^\top$ must be one-to-one, because $\beta \mapsto \beta \cdot T$ is one-to-one. Set $B := A^\top (AA^\top)^{-1} \in \mathbb{R}^{n \times 2k}$. Then $AB = I$. Set $E_0 := \mathbb{T} \setminus \{\mathrm{e}^{-i3\pi/4}\}$. Let $\arg : E_0 \to (-3\pi/4, 5\pi/4)$ denote a continuous branch of arg and also the corresponding function $E_0^{2k} \to (-3\pi/4, 5\pi/4)^{2k}$. Define $\phi \in C(E_0^{2k}; \mathbb{T}^n)$ by $\phi(z) := \mathrm{e}^{iB \arg z}$. Define the function $h \in C(E_0^{2k}; \mathbb{C}^k)$ by

$$h(z) := \sum_{\alpha \in \mathbb{Z}^n} h_\alpha \phi(z)^\alpha \quad (z \in E_0^{2k}).$$

[4] By $\top$ we denote the transpose, by T the vector in (11).

By Lemma 5.9, there exists $z \in E_{2k} \subset E_0^{2k}$ such that $(f + gh)(z) = 0$. Set $x := \arg z$.

Let $\epsilon > 0$ be given. Using the uniform continuity of f and g one can show that there exists $\delta' > 0$ such that

$$|f(z') + g(z')c| < \epsilon \quad \text{when } z \in \mathbb{T}^{2k},\ c \in \mathbb{C}^k,\ |z' - z| < \delta',\ |c - h(z)| < \delta'. \tag{12}$$

Set $h''(w) := \sum_{\alpha \in \mathbb{Z}^n} h_\alpha w^\alpha$ ($w \in \mathbb{T}^n$). Pick $\delta \in (0, \delta')$ such that

$$|h''(w) - h''(w')| < \delta' \quad \text{when } w, w' \in \mathbb{T}^n,\ |w - w'| < \delta. \tag{13}$$

By Kronecker's Theorem, there exists $s \in i\mathbb{R}$ such that $|\mathrm{e}^{-sr} - \mathrm{e}^{iBx}|$ is arbitrarily small, hence such that

$$|sr + iBx + i2\pi q| < \delta/|A| \le \delta \quad \text{and hence} \quad \left|\mathrm{e}^{-sr} - \mathrm{e}^{iBx}\right| < \delta$$

for some $q \in \mathbb{Z}^n$. Then $\delta > |sAr + iABx + i2\pi Aq| = |st + ix + i2\pi Aq|$. Therefore,

$$\left|\mathrm{e}^{-sT} - \mathrm{e}^{ix}\right| = \left|\mathrm{e}^{-sT} - \mathrm{e}^{i(x+2\pi Aq)}\right| < \delta. \tag{14}$$

Since $z = \mathrm{e}^{ix}$, we conclude from (12)–(14) that (set $z' := \mathrm{e}^{-sT}$, $w := \mathrm{e}^{-sr}$, and $w' := \mathrm{e}^{iBx}$, and note that $h'(s) = h''(\mathrm{e}^{-sr}) =: c$ and $h(z) = h''(\phi(z)) = h''(\mathrm{e}^{iBx})$)

$$|f'(s) + g'(s)h'(s)| = |f(\mathrm{e}^{-sT}) + g(\mathrm{e}^{-sT})h''(\mathrm{e}^{-sr})| = |f(z') + g(z')c| < \epsilon. \tag{15}$$

As $\epsilon > 0$ was arbitrary and $s \in i\mathbb{R}$, the proof is complete. □

Observe that $\mathcal{F}(\mathcal{M}_+^{(n^+)})_{\mathbb{R}}$ consists of functions of the form $\sum_{\alpha \in \mathbb{N}^n} a_\alpha \mathrm{e}^{-\cdot \sum_{j=1}^n \alpha_j T_j}$, where $a \in \ell^1(\mathbb{N}^n; \mathbb{R})$.

Corollary 6.3. *Let $T_1, T_2, \ldots > 0$ be $\mathbb{Q}$-linearly independent. We have* $\operatorname{bsr} \mathcal{A} \ge \lfloor n/2 \rfloor + 1$ *when $\mathcal{A} \subset \mathrm{AP}$ is any (real or complex) function algebra that contains $\mathcal{F}(\mathcal{M}_+^{(n^+)})_{\mathbb{R}}$.*

In particular, $\operatorname{bsr} \mathcal{A} = \infty$ *when $\mathcal{A}$ stands for any of the following: $\mathcal{M}$, $\mathcal{M}_{\mathbb{R}}$, $\mathcal{M}_+$, $\mathcal{M}_{+\mathbb{R}}$, AP, $(\mathrm{AP})_{\mathbb{R}}$, AP_+, $(\mathrm{AP}_+)_{\mathbb{R}}$.*

Recall that tsr $\ge$ bsr, by Proposition 3.2. The assumption on $\mathcal{F}(\mathcal{M}_+^{(n^+)})_{\mathbb{R}}$ will be weakened in Lemma 9.7.

Proof. If $k := \lfloor n/2 \rfloor \ge 1$, define f', g' as in Lemma 6.2. If $h' \in \mathcal{A} \subset \mathrm{AP}$, then, for every $F \in \mathcal{A}$ we have $\inf_{i\mathbb{R}} |F \cdot (f' + g'h')| = 0$. Hence $F \cdot (f' + g'h') \ne 1$, and so $f' + g'h' \notin \mathrm{U}_k(\mathcal{A})$. Consequently, $\operatorname{bsr} \mathcal{A} \ge k+1$. Since k was arbitrary, $\operatorname{bsr} \mathcal{A} = \infty$. The second paragraph follows from the first (because $\mathcal{F}$ is an isomorphism). □

The result $\operatorname{bsr} \mathrm{AP} = \infty$ was already established in [33], using an alternative method (and the fact that tsr = bsr for C^*-algebras [14]). The other results seem to be new.

Corollary 6.4. *We have* $\operatorname{bsr} \mathcal{A} = \infty$ *when $\mathcal{A}$ stands for any of the following $\ell^1(\mathbb{R})$, $\ell^1(\mathbb{R}; \mathbb{R})$, $\ell^1(\mathbb{R}_+)$, $\ell^1(\mathbb{R}_+; \mathbb{R})$, $\mathcal{M} + \mathrm{L}^1$, $(\mathcal{M} + \mathrm{L}^1)_{\mathbb{R}}$, $\mathcal{M}_+ + \mathrm{L}^1_+$, and $(\mathcal{M}_+ + \mathrm{L}^1_+)_{\mathbb{R}}$.*

Proof. For the ℓ^1 classes, use Corollary 6.3 and Remark 2.1. By Lemma 3.9 we get the other four results (because, for example, $(\mathcal{M} + \mathrm{L}^1)/\mathrm{L}^1 = \mathcal{M}$). □

Remark 6.5 (Answer to Mortini's question)**.** S. Treil showed that $\operatorname{bsr} H^\infty = 1$ [36]. The question about what the situation looks like for subalgebras of H^∞ was asked in [20], where it was shown that H^∞ has closed subalgebras with arbitrary stable rank, and even bsr $= \infty$

is possible. However, in these subalgebras constructed in [20], the unit disk is not dense in the spectrum, and the following open problem was mentioned there:

> Is there a closed subalgebra $\mathcal{A}$ of H^∞, such that the maximal ideal space of $\mathcal{A}$ contains $\mathbb{D}$ as a dense subset, and $\operatorname{bsr}\mathcal{A} > 1$?

We have answered this question affirmatively (with the isomorphic space $H^\infty(\mathbb{C}_+)$ in place of $H^\infty(\mathbb{D})$). Indeed, AP_+ is a closed subalgebra of the Hardy space $H^\infty(\mathbb{C}_+)$ of the half-plane, and its maximal ideal space contains the open half-plane $\mathbb{C}_+$ as a dense subset [24, Theorem 2.3]. Since we have shown that $\operatorname{bsr}\mathrm{AP}_+ = \infty$ in Corollary 6.3, this settles the above question.

If $n \in \{2, 3, \ldots\}$, then the algebra $\mathcal{A}_n := \mathrm{AP}_+^{2n-2} \subset H^\infty(\mathbb{C}_+)$ of Remark 8.6 below satisfies $\operatorname{bsr}\mathcal{A}_n = n$, and $\mathbb{C}_+$ is dense in the maximal ideal space of $\mathcal{A}_n$. Thus, the requirement that $\mathbb{C}_+$ is dense in the maximal ideal space (of a closed subalgebra $\mathcal{A}$ of H^∞) does not exclude any value of $\operatorname{bsr}\mathcal{A}$.

7. Mixed measures $\mathcal{M}_*^* + \mathrm{L}^1$

In this section we study the algebras of the type $\mathcal{A}+\mathrm{L}^1$, that is, we take a (discrete measure) subalgebra $\mathcal{A}$ of $\mathcal{M}$ and add an absolutely continuous (L^1) part.

First we establish some technical results on unimodularity and zeros, and a corona theorem (Corollary 7.4). In Theorems 7.6 and 7.7 we show that $\operatorname{bsr}(\mathcal{A} + \mathrm{L}^1) = \operatorname{bsr}\mathcal{A}$ etc. in typical cases. Finally, we prove Theorem 1.2. As always, we accompany any complex results by corresponding real results.

We start by showing that the discrete part of a left inverse of a measure can be made equal to any left inverse of the discrete part. (Here $G \cdot F := \sum_{k=1}^n G_k * F_k$.)

Lemma 7.1. *Let $F, G \in \mathcal{M}^n$, $G \cdot F = 1$, $f \in \mathrm{L}^1(\mathbb{R})^n$ and $F + f \in \mathrm{U}_n(\mathcal{M} + \mathrm{L}^1)$. Then there exists $g \in \mathrm{L}^1(\mathbb{R})^n$ such that $(G + g) \cdot (F + f) = 1$.*

If F, G, f have their supports on $\mathbb{R}_+$, then we can have $g \in \mathrm{L}^1(\mathbb{R}_+)^n$. In addition, if F, G, f are real-valued, then we can have g real-valued.

Proof.

1° Replace all measures/functions by their Laplace transforms. Since $G\cdot f \in \mathcal{F}\mathrm{L}^1$, by the Riemann–Lebesgue Lemma there exists $R < \infty$ such that $|(1 + G \cdot f)(z)| \geq \epsilon > 0$ for $|z| > R$. With $\phi(z) := (z+1)^{-1} \in \mathcal{F}\mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$, $|(1+G\cdot f, \phi)| \geq \min\{\epsilon, 1/(R+1)\} > 0$. By Theorem 4.3, this means that $(1 + G \cdot f, \phi) \in \mathcal{F}\mathrm{U}_2(\mathbb{C}\delta_0 + \mathrm{L}^1)$.

From Corollary 5.21, $\operatorname{bsr}\mathbb{C}\delta_0 + \mathrm{L}^1 = 1$. So there exists $h \in \mathcal{F}\mathbb{C}\delta_0 + \mathrm{L}^1$ such that $h_1 := 1+G\cdot f+h\phi \in \mathcal{F}\mathrm{U}_1(\mathbb{C}\delta_0+\mathrm{L}^1)$. Pick $w \in \mathcal{F}(\mathcal{M}+\mathrm{L}^1)^n$ such that $w\cdot(F+f) = 1$. With $g_1 := h\phi w \in (\mathcal{F}\mathrm{L}^1)^n$, $(G+g_1)\cdot(F+f) = 1+G\cdot f+h\phi w\cdot(F+f) = h_1$. By Lemma 4.1, we have $h_1^{-1} = 1+h_2 \in \mathcal{F}\mathbb{C}\delta_0+\mathrm{L}^1$ for some $h_2 \in \mathcal{F}\mathrm{L}^1$, and so $h_1^{-1}(G+g_1) = G+g$, where $g := h_2G + g_1 + h_2g_1 \in (\mathcal{F}\mathrm{L}^1)^n$.

2° Now we consider the original measures instead of their transforms. If F, G, f have their supports on $\mathbb{R}_+$, then so have the measures in 1°. If F and f are real-valued, we can replace $G + g$ by $G_R + g_r$ (by Lemma 4.4); if G is real-valued, then $G = G_R$.

□

Next we note that if the discrete part of a measure is unimodular (left-invertible), then so is the whole measure iff its transform has no "finite" zeros.

Lemma 7.2. *Let $\mu \in \mathcal{M}^n$ and $f \in \mathrm{L}^1(\mathbb{R})^n$. Then $\mu + f \in \mathrm{U}_n(\mathcal{M} + \mathrm{L}^1)$ iff $\mu \in \mathrm{U}_n(\mathcal{M})$ and $\widehat{\mu} + \widehat{f} \neq 0$ on $i\mathbb{R}$.*

Let $\mu \in (\mathcal{M}_+)^n$ and $f \in \mathrm{L}^1(\mathbb{R}_+)^n$. Then $\mu + f \in \mathrm{U}_n(\mathcal{M}_+ + \mathrm{L}^1_+)$ iff $\mu \in \mathrm{U}_n(\mathcal{M}_+)$ and $\widehat{\mu} + \widehat{f} \neq 0$ on $\overline{\mathbb{C}_+}$.

Proof. We prove the first equivalence; the proof of the second one is analogous.

By Lemma 4.1 and Theorem 4.3, the "only if" claim holds. Assume then that $\mu \in \mathrm{U}_n(\mathcal{M})$ and $\widehat{\mu} + \widehat{f} \neq 0$ on $i\mathbb{R}$. By Theorem 4.3, $\epsilon := \inf_{i\mathbb{R}} |\widehat{\mu}| > 0$.

From the Riemann–Lebesgue Lemma, there exists $R < \infty$ such that $|f(z)| < \epsilon/2$ when $|z| > R$. But $\delta := \inf_{|z| \leq R} |\widehat{\mu} + \widehat{f}| > 0$, and so $|\widehat{\mu} + \widehat{f}| \geq \min\{\delta, \epsilon/2\}$ everywhere. It follows from Theorem 4.3 that $\mu + f \in \mathrm{U}_n(\mathcal{M} + \mathrm{L}^1)$. □

Now we extend the above result to $\mathcal{A}$ in place of $\mathcal{M}$ (and include the real case).

Lemma 7.3. *Assume that $\mathcal{A}$ is a subalgebra of $\mathcal{M}$ and that $\bar{F} \in \mathcal{A}$ for every $F \in \mathcal{A}$ (or drop the claims on $\mathcal{A}_\mathbb{R}$, $\mathcal{M}_\mathbb{R}$ and $\mathcal{M}_{+\mathbb{R}}$ below). Then $\mathcal{A}' := \mathcal{A} + \mathrm{L}^1(\mathbb{R})$ is a subalgebra of $\mathcal{M} + \mathrm{L}^1(\mathbb{R})$, and $\mathrm{U}_n(\mathcal{A}'_\mathbb{R}) = \mathcal{A}'_\mathbb{R} \cap \mathrm{U}_n(\mathcal{A}')$.*

Let $\mu \in \mathcal{A}^n$ and $f \in \mathrm{L}^1(\mathbb{R})^n$. Then $\mu + f \in \mathrm{U}_n(\mathcal{A} + \mathrm{L}^1(\mathbb{R}))$ iff $\widehat{\mu} + \widehat{f} \neq 0$ on $i\mathbb{R}$ and $\mu \in \mathrm{U}_n(\mathcal{A})$, or equivalently, iff $\mu + f \in \mathrm{U}_n(\mathcal{M} + \mathrm{L}^1(\mathbb{R}))$ and $\mu \in \mathrm{U}_n(\mathcal{A})$.

The above also holds if we replace $\mathcal{M}$ by $\mathcal{M}_+$ (respectively, $\mathcal{M}_\mathbb{R}$, $\mathcal{M}_{+\mathbb{R}}$), $\mathrm{L}^1(\mathbb{R})$ by $\mathrm{L}^1(\mathbb{R}_+)$ (respectively, $\mathrm{L}^1(\mathbb{R};\mathbb{R})$, $\mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$), and $i\mathbb{R}$ by $\overline{\mathbb{C}_+}$ (respectively, $i\mathbb{R}$, $\overline{\mathbb{C}_+}$).

By Lemma 4.2, this shows that $X(\mathcal{A}) \cup i\mathbb{R}$ is dense in $X(\mathcal{A} + \mathrm{L}^1(\mathbb{R}))$.

Proof. Obviously $\mathcal{A} + \mathrm{L}^1$ is a subalgebra of $\mathcal{M}$ (it is a Banach algebra iff $\mathcal{A}$ is complete). The claim $\mathrm{U}_n(\mathcal{A}'_\mathbb{R}) = \mathcal{A}'_\mathbb{R} \cap \mathrm{U}_n(\mathcal{A}')$ follows by taking the real part of a left inverse.

If $\mu + f \in \mathrm{U}_n(\mathcal{A}')$ (that is, $(G+g) \cdot (\mu + f) = 1$ for some $G + g \in (\mathcal{A}')^n \subset (\mathcal{M} + \mathrm{L}^1)^n$), then from Theorem 4.3 we get $\inf_{i\mathbb{R}} |\widehat{\mu} + \widehat{f}| > 0$, and $G \cdot \mu = I$, so $\mu \in \mathrm{U}_n(\mathcal{A})$.

Conversely, if $\mu \in \mathrm{U}_n(\mathcal{A})$ (that is, $G \cdot \mu = 1$ for some $G \in \mathcal{A}$), and $\widehat{\mu} + \widehat{f} \neq 0$ on $i\mathbb{R}$, then from Lemma 7.2, $\mu + f \in \mathrm{U}_n(\mathcal{M} + \mathrm{L}^1)$. By Lemma 7.1, $(G+g) \cdot (\mu + f) = 1$ for some $g \in \mathrm{L}^1(\mathbb{R})^n$, and so $\mu + f \in \mathrm{U}_n(\mathcal{A}')$.

The above proof also applies in the three other cases (use Theorem 4.5 for real-valued measures). □

If the corona theorem holds for some subalgebra $\mathcal{A}$ of $\mathcal{M}$, then it also holds for $\mathcal{A} + \mathrm{L}^1$ (that is, $i\mathbb{R}$ is then also dense in $X(\mathcal{A} + \mathrm{L}^1)$), etc., as shown below.

Corollary 7.4 ($\mathcal{A} + \mathrm{L}^1$ corona)**.** *Let $\mathcal{A} \subset \mathcal{M}$ be a n-full (that is, $\mathrm{U}_n(\mathcal{A}) = \mathcal{A}^n \cap \mathrm{U}_n(\mathcal{M})$) subalgebra of $\mathcal{M}$, and that $\bar{F} \in \mathcal{A}\ \forall F \in \mathcal{A}$ (or drop the claims on $\mathcal{A}_\mathbb{R}$). Then $\mathrm{U}_n(\mathcal{A} + \mathrm{L}^1) = (\mathcal{A} + \mathrm{L}^1)^n \cap \mathrm{U}_n(\mathcal{M} + \mathrm{L}^1)$, and $\mathrm{U}_n(\mathcal{A}_\mathbb{R} + \mathrm{L}^1(\mathbb{R};\mathbb{R})) = (\mathcal{A}_\mathbb{R} + \mathrm{L}^1(\mathbb{R};\mathbb{R}))^n \cap \mathrm{U}_n(\mathcal{M} + \mathrm{L}^1)$, hence then the Corona Theorem 4.3(a1) holds for $\mathcal{A}$, $\mathcal{A}_\mathbb{R}$, $\mathcal{A} + \mathrm{L}^1$ and $\mathcal{A}_\mathbb{R} + \mathrm{L}^1(\mathbb{R};\mathbb{R})$ (for this n).*

The above also holds with (a1), $\mathcal{M}$, L^1, $\mathrm{L}^1(\mathbb{R};\mathbb{R})$ and $i\mathbb{R}$ replaced by (a2), $\mathcal{M}_+$, L^1_+, $\mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$ and $\mathbb{C}_+$, respectively.

Proof. The first two claims follow from Lemmata 7.3 and 4.1 and n-fullness. Then the corona claim follows from Theorem 4.3 (applied to $\mathcal{M}$ and for $\mathcal{M}+\mathrm{L}^1$). The other cases are analogous (Theorem 4.5). □

By *polynomials of atoms* we mean finite linear combinations of δ_r ($r \in \mathbb{R}$). The Laplace transform of a polynomial of atoms is of the form $\sum_{k=1}^n a_k \mathrm{e}^{-\cdot r_k}$, and so entire.

We denote $\mathcal{M}$ (respectively, $\mathcal{M}_+$) by $(\mathcal{M})_{\mathbb{K}}$ (respectively, $(\mathcal{M}_+)_{\mathbb{K}}$) if $\mathbb{K} = \mathbb{C}$. Note that $\mathbb{K}+\mathrm{L}^1(\mathbb{R};\mathbb{K}) = (\mathbb{C}\delta_0+\mathrm{L}^1)_{\mathbb{K}}$. Next we show that the "finite" zeros of an element of $\mathcal{M}+\mathrm{L}^1$ on $i\mathbb{R}$ or of an element of $(\mathcal{M}_+ + \mathrm{L}^1_+)^n$ on $\overline{\mathbb{C}_+}$ can be removed by an arbitrarily small perturbation.

Lemma 7.5. *If $\mathcal{F}^{-1}F \in (\mathcal{M})_{\mathbb{K}}$ is a polynomial of atoms, $f \in \mathcal{F}\mathrm{L}^1(\mathbb{R};\mathbb{K})$, and $\epsilon > 0$, then there exists $g \in \mathbb{K}1 + \mathcal{F}\mathrm{L}^1(\mathbb{R};\mathbb{K})$ such that $\|g\| < \epsilon$ and $F + f + g$ has no zeros on $i\mathbb{R}$.*

If $n \geq 2$, $F \in \mathcal{F}(\mathcal{M}_+)^n_{\mathbb{K}}$, $\mathcal{F}^{-1}F_2$ is a polynomial of atoms, $f \in \mathcal{F}\mathrm{L}^1(\mathbb{R}_+;\mathbb{K})^n$, and $\epsilon > 0$, then there exists $c \in \mathbb{R}$, $h \in \mathcal{F}\mathrm{L}^1(\mathbb{R}_+;\mathbb{K})$ such that $\|(c,h)\| < \epsilon$ and $F + f + g$ has no zeros on $\overline{\mathbb{C}_+}$, where $g := (c, h, 0, \cdots, 0)$.

Proof. By density, in 1° and 2° we can replace f by $\hat{w}$, where w is continuous with compact support (in the end, set $g := c + \hat{w} - f$) so that f becomes entire; similarly, replace f_2 by $\hat{w}$ in 3°.

1° Case $\mathbb{K} = \mathbb{C}$, $\mathcal{F}^{-1}F \in \mathcal{M}$. The set $E := (F + f)[i\mathbb{R}] \subset \mathbb{C}$ has zero measure, by Lemma A.2 (because F and f are holomorphic, hence C^1). Pick any $c \in -E^c$ such that $F + f + c \neq 0$ on $i\mathbb{R}$.

2° Case $\mathbb{K} = \mathbb{R}$, $\mathcal{F}^{-1}F \in (\mathcal{M})_{\mathbb{R}}$. Now $f(ir) = \int_{\mathbb{R}} \mathrm{e}^{-irt} w(t)\,dt$ for some $w \in C(\mathbb{R};\mathbb{R})$ with compact support. Moreover, $F(z) = \sum_{k=1}^m a_k \mathrm{e}^{-zT_k}$ for some m, a_k, T_k. Therefore,

$$G(r) := -\operatorname{Im}(F + f)(ir) = \sum_{k=1}^m a_k \sin rT_k + \int_{\mathbb{R}} w(t) \sin rt\,dt \quad (r \in \mathbb{R}).$$

Since G is the restriction to $\mathbb{R}$ of an entire function, the zeros of G are isolated. So $\operatorname{Im}(F + f)$ has a countable number of zeros on $i\mathbb{R}$, that is, $E := (F + f)[i\mathbb{R}] \cap \mathbb{R}$ is countable. Let $-c \in (-\epsilon, \epsilon) \setminus E$. Then $F + f + c$ has no zeros on $i\mathbb{R}$.

3° Case $\mathcal{F}^{-1}F \in \mathcal{M}_+$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Since $F_2 + f_2$ is holomorphic, the set $Z := \{z \in \overline{\mathbb{C}_+} : (F_2 + f_2)(z) = 0\}$ is countable. Let $E := (F_1 + f_1)[Z]$, and $-c \in (-\epsilon, \epsilon) \setminus E$. Then $F_1 + f_1 + c$ has no zeros on Z, hence $F + f + (c, 0, 0, \cdots, 0)$ has no zeros on $\overline{\mathbb{C}_+}$. □

In Lemma 7.5, the function $F + f + g$ might still have zeros at some other points of the maximal ideal space (of some relevant algebra, such as $\mathcal{M}+\mathrm{L}^1$). However, that cannot happen if F is unimodular, by Lemma 7.3.

In our examples of algebras $\mathcal{A} \subset \mathcal{M}$, the polynomials of atoms contained in $\mathcal{A}$ are dense in $\mathcal{A}$, so the following proposition can be applied. This shows that enlarging $\mathcal{A}$ by L^1 functions does not increase $\operatorname{tsr}\mathcal{A}$.

Theorem 7.6. *Assume that $\mathcal{A}$ is a* Banach subalgebra[5] *of $\mathcal{M}$ and that the polynomials of atoms contained in $\mathcal{A}$ are dense in $\mathcal{A}$.*

Then $\operatorname{tsr}(\mathcal{A} + \mathrm{L}^1(\mathbb{R})) = \operatorname{tsr}\mathcal{A}$. *If $\mathcal{A} \subset \mathcal{M}_+$, then* $\operatorname{tsr}(\mathcal{A} + \mathrm{L}^1(\mathbb{R}_+)) = \max\{\operatorname{tsr}\mathcal{A}, 2\}$.

Assume, instead, that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{M}_{\mathbb{R}}$, and that the polynomials of atoms contained in $\mathcal{A}$ are dense in $\mathcal{A}$.

Then $\operatorname{tsr}(\mathcal{A} + \mathrm{L}^1(\mathbb{R};\mathbb{R})) = \operatorname{tsr}\mathcal{A}$. *If $\mathcal{A} \subset \mathcal{M}_{+\mathbb{R}}$, then* $\operatorname{tsr}(\mathcal{A} + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})) = \max\{\operatorname{tsr}\mathcal{A}, 2\}$.

Proof.

0° If $n := \operatorname{tsr}(\mathcal{A} + \mathrm{L}^1)$, $\mu \in \mathcal{A}^n$ and $\epsilon > 0$, then $\epsilon > \|\mu' + f - \mu\| = \|\mu' - \mu\| + \|f\|$ for some $\mu' + f \in \mathrm{U}_n(\mathcal{A} + \mathrm{L}^1)$; by the proof of Lemma 4.1, then $\mu' \in \mathrm{U}_n(\mathcal{A})$, hence then

[5] That is, $\mathcal{A}$ is a closed subalgebra of $\mathcal{M}$ with the same norm.

$\operatorname{tsr}\mathcal{A} \le n$. Thus, $\operatorname{tsr}\mathcal{A} \le \operatorname{tsr}(\mathcal{A}+\mathrm{L}^1)$. Similarly, we get the other four "$\ge$" signs (use, e.g., Lemma 5.11 for the two ≥ 2 results).

1° The claim $\operatorname{tsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R})) \le \operatorname{tsr}\mathcal{A}$. Set $n := \operatorname{tsr}\mathcal{A}$. Let $F \in \mathcal{F}\mathcal{A}^n$, $f \in (\mathcal{F}\mathrm{L}^1)^n$, $\epsilon > 0$ be given. Define $\mathcal{F}\mathcal{A}' := \mathcal{F}(\mathcal{A}+\mathrm{L}^1)$. We shall show that $F + f \in \overline{\mathrm{U}_n(\mathcal{F}\mathcal{A}')}$ (and so $\operatorname{tsr}\mathcal{F}\mathcal{A}' \le n$). Since $\operatorname{tsr}\mathcal{A} \le n$, by density we can assume that $F \in \mathrm{U}_n$ (that is, $G \cdot F = 1$ for some $G \in \mathcal{F}\mathcal{A}^n$); we can simultaneously assume that F is a polynomial of atoms, by density and Lemma 3.4 (and n-fullness, if $\mathcal{A}$ is not complete).

By Lemma 7.5, there exists $g \in (\mathcal{F}\mathcal{A}')^n$ such that $F + f + g$ has no zeros on $i\mathbb{R}$. From Lemma 7.3, it follows that $F + f + g \in \mathrm{U}_n(\mathcal{A}+\mathrm{L}^1)$.

2° The claim $\operatorname{tsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R};\mathbb{R})) \le \operatorname{tsr}\mathcal{A}$. The above proof still applies.

3° The claims $\operatorname{tsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R}_+)) \le \max\{\operatorname{tsr}\mathcal{A}, 2\}$ and $\operatorname{tsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R}_+;\mathbb{R})) \le \max\{\operatorname{tsr}\mathcal{A}, 2\}$. Let $n := \max\{\operatorname{tsr}\mathcal{A}, 2\}$ and work as in 1° (recall Lemma 3.1).

□

Almost analogous results hold for Bass stable ranks as well.

Theorem 7.7. *Assume that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{M}$, and that the polynomials of atoms contained in $\mathcal{A}$ are dense in $\mathcal{A}$.*

Then $\operatorname{bsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R})) = \operatorname{bsr}\mathcal{A}$. *If $\mathcal{A} \subset \mathcal{M}_+$, then* $\operatorname{bsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R}_+)) = \operatorname{bsr}\mathcal{A}$.

Assume, instead, that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{M}_{\mathbb{R}}$, and that the polynomials of atoms contained in $\mathcal{A}$ are dense in $\mathcal{A}$.

Then $\operatorname{bsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R};\mathbb{R})) = \operatorname{bsr}\mathcal{A}$. *If $\mathcal{A} \subset \mathcal{M}_{+\mathbb{R}}$, then* $\operatorname{bsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R}_+;\mathbb{R})) = \max\{\operatorname{bsr}\mathcal{A}, 2\}$.

Proof.

0° We have $\operatorname{bsr}(\mathcal{A}+\mathrm{L}^1) \ge \max\{\operatorname{bsr}\mathcal{A}, \operatorname{bsr}\mathbb{C}\delta_0 + \mathrm{L}^1\}$, by Lemma 3.9. Therefore, only $\operatorname{bsr}(\mathcal{A}+\mathrm{L}^1) \le \operatorname{bsr}\mathcal{A}$ needs to be proved. Similarly, from Corollary 5.22 it follows that when $\mathcal{A} \subset \mathcal{M}_+$, we have $\operatorname{bsr}(\mathcal{A}+\mathrm{L}^1_+) \ge \max\{\operatorname{bsr}\mathcal{A}, \operatorname{bsr}\mathbb{C}\delta_0 + \mathrm{L}^1_+\} = \max\{\operatorname{bsr}\mathcal{A}, 2\}$, so also then only the converses need to be proved. Analogous claims hold for the real cases too.

1° We prove that $\operatorname{bsr}(\mathcal{A}+\mathrm{L}^1(\mathbb{R})) \le \operatorname{bsr}\mathcal{A}$. Set $n := \operatorname{bsr}\mathcal{A}$, $\mathcal{F}\mathcal{A}' := \mathcal{F}(\mathcal{A}+\mathrm{L}^1(\mathbb{R}))$. Let $F \in \mathcal{F}\mathcal{A}^n$, $f \in \mathcal{F}\mathrm{L}^1(\mathbb{R})^n$, $G \in \mathcal{F}\mathcal{A}$ and $g \in \mathcal{F}\mathrm{L}^1(\mathbb{R})$ be such that

$$(a, b) := (F + f, G + g) \in \mathrm{U}_{n+1}(\mathcal{F}\mathcal{A}').$$

It suffices to find $H + h \in \mathcal{F}(\mathcal{A}+\mathrm{L}^1(\mathbb{R}))^n$ such that $a + b(H + h) \in \mathcal{F}\mathrm{U}_n(\mathcal{A}+\mathrm{L}^1(\mathbb{R}))$.

By Lemma 7.3, $(F, G) \in \mathrm{U}_{n+1}(\mathcal{F}\mathcal{A})$. As $\operatorname{bsr}\mathcal{A} \le n$, there exists $H \in \mathcal{F}\mathcal{A}^n$ such that $w := F + GH \in \mathrm{U}_n(\mathcal{F}\mathcal{A})$. Set $v := a + bH = w + f + gH \in \mathcal{F}\mathcal{A}^n$. Then $(v, b) \in \mathrm{U}_{n+1}(\mathcal{F}\mathcal{A}')$, by Lemma 3.10. If $f' := f + gH$, then $v = w + f'$. From Lemma 7.3, it follows that $(v, b\phi) \in \mathrm{U}_{n+1}(\mathcal{F}\mathcal{A}')$, where $\phi(z) := (1 + z)^{-1}$. Since $\mathcal{F}\mathcal{A}'$ is commutative, this means that there exist $w' + g' \in (\mathcal{F}\mathcal{A}')^n$ (with $w' \in \mathcal{F}\mathcal{A}^n$, $g' \in \mathcal{F}\mathrm{L}^1(\mathbb{R})^n$) and $b' \in \mathcal{F}\mathcal{A}'$ such that $1 = v \cdot (w' + g') + b\phi b'$. As $\phi \in \mathcal{F}\mathrm{L}^1$, this implies that $w \cdot w' = 1$, by (the proof of) Lemma 4.1, and so $w' \in \mathrm{U}_n(\mathcal{F}\mathcal{A})$.

Let $\mathcal{F}^{-1}u \in \mathrm{U}_n(\mathcal{A})$ be a polynomial of atoms (we can have u arbitrarily close to w'). By Lemma 7.5, there exists $g'' \in (\mathbb{C}1 + \mathcal{F}\mathrm{L}^1(\mathbb{R}))^n$ such that $u + g''$ is arbitrarily close to $u + g'$ but has no zeros, and so $x := u + g'' \in \mathrm{U}_n(\mathcal{F}\mathcal{A}')$, from Lemma 7.3. With x sufficiently close to $w' + g'$ we have $\mathrm{U}_1(\mathcal{F}\mathcal{A}') \ni v \cdot x + b\phi b' =: y$. Pick $x' \in \mathrm{U}_n(\mathcal{F}\mathcal{A}')$ such that $x' \cdot x = 1$. Set $h := \phi b' x'$. Then

$$(v + bh) \cdot xy^{-1} = (v \cdot x + b\phi b' x' \cdot x)y^{-1} = yy^{-1} = 1,$$

and so $v + bh \in \mathrm{U}_n(\mathcal{FA}')$. But $v + bh = a + b(H + h)$ and $H + h \in (\mathcal{FA}')^n$. Since $(a, b) \in \mathrm{U}_{n+1}(\mathcal{FA}')$ was arbitrary, $\operatorname{bsr} \mathcal{FA}' \le n$.

2° Part 1°, mutatis mutandis, also establishes the inequalities $\operatorname{bsr}(\mathcal{A} + \mathrm{L}^1(\mathbb{R}; \mathbb{R})) \le \operatorname{bsr} \mathcal{A}$ and $\operatorname{bsr}(\mathcal{A} + \mathrm{L}^1(\mathbb{R}_+; \mathbb{K})) \le n$ for $n := \max\{\operatorname{bsr} \mathcal{A}, 2\}$, under corresponding assumptions. Thus it remains to assume that $\mathcal{A} \subset \mathcal{M}_+$ and $\operatorname{bsr} \mathcal{A} = 1$ and to show that $\operatorname{bsr} \mathcal{A}' = 1$, where $\mathcal{A}' := \mathcal{A} + \mathrm{L}^1(\mathbb{R}_+)$.

To that end, we work as in 1° with $n := 1$ to observe that $(v, b\phi) \in \mathrm{U}_2(\mathcal{FA}')$, where $v = w + f'$ (but we cannot apply Lemma 7.5). Note that $\mathbb{C}\delta_0 + \mathrm{L}^1_+ \subset \mathcal{A}'$. By Lemma 3.10, we have $(1 + w^{-1} f', b\phi) \in \mathrm{U}_2(\mathcal{FA}')$. But $(1 + w^{-1} f', b\phi) \in (\mathbb{C} + \mathcal{F}\mathrm{L}^1(\mathbb{R}_+))^2$. Apply Lemma 7.3 to $\mathcal{A}'$ and to $\mathbb{C}\delta_0 + \mathrm{L}^1(\mathbb{R}_+)$ to get that $(1 + w^{-1} f', b\phi) \in \mathrm{U}_2(\mathbb{C} + \mathcal{F}\mathrm{L}^1(\mathbb{R}_+))$.

Since $\operatorname{bsr}(\mathbb{C}\delta_0 + \mathrm{L}^1_+) = 1$, by Corollary 5.22, there exists $h_0 \in \mathcal{F}\mathbb{C}\delta_0 + \mathrm{L}^1_+$ such that $x := 1 + w^{-1} f' + b\phi h_0 \in \mathcal{F}\mathrm{U}_1(\mathbb{C}\delta_0 + \mathrm{L}^1_+)$. So $(w + f' + b\phi h_0 w) w^{-1} x^{-1} = 1$. If $h := \phi h_0 w$, then $v + bh \in \mathrm{U}_1$. Hence $\operatorname{bsr} \mathcal{A}' \le 1$, as in 1°.

□

Remark 7.8. Instead of $\mathcal{A}$ being a Banach subalgebra of $\mathcal{M}$, in Theorems 7.6 and 7.7 it suffices to assume that $\mathcal{A}$ is a topological subring of $\mathcal{M}$ (with the inherited or even weaker topology) and that $\mathrm{U}_n(\mathcal{A})$ is open for every $n \ge 1$ (this last condition is satisfied if $\mathcal{A}$ is n-full in some Banach subalgebra of $\mathcal{M}$ or of $\mathcal{M}_\mathbb{R}$).

(Use [37, Theorem 4] instead of Lemma 3.9 in the proof of Theorem 7.7.)

From Theorems 7.6 and 7.7 and Corollaries 5.14 and 5.17, we get the following.

Corollary 7.9. *We have* $\operatorname{bsr}(\mathcal{M}^{(n)} + \mathrm{L}^1) = \operatorname{bsr} \mathcal{M}^{(n)} = \lfloor n/2 \rfloor + 1 = \operatorname{tsr}(\mathcal{M}^{(n)} + \mathrm{L}^1) = \operatorname{tsr} \mathcal{M}^{(n)}$, $\operatorname{bsr}(\mathcal{M}^{(n)} + \mathrm{L}^1)_\mathbb{R} = \operatorname{bsr}(\mathcal{M}^{(n)})_\mathbb{R} = \lfloor n/2 \rfloor + 1 = \operatorname{tsr}(\mathcal{M}^{(n)} + \mathrm{L}^1)_\mathbb{R} = \operatorname{tsr}(\mathcal{M}^{(n)})_\mathbb{R}$, *and* $\operatorname{bsr}(\mathcal{M}^{(n^+)}_+ + \mathrm{L}^1_+) = \operatorname{bsr} \mathcal{M}^{(n^+)}_+ = \lfloor n/2 \rfloor + 1$, $\operatorname{tsr}(\mathcal{M}^{(n^+)}_+ + \mathrm{L}^1_+) = \operatorname{tsr} \mathcal{M}^{(n^+)}_+ = n + 1$, $\operatorname{bsr}(\mathcal{M}^{(n^+)}_+ + \mathrm{L}^1_+)_\mathbb{R} = \operatorname{bsr}(\mathcal{M}^{(n^+)}_+)_\mathbb{R} = n + 1 = \operatorname{tsr}(\mathcal{M}^{(n^+)}_+ + \mathrm{L}^1_+)_\mathbb{R} = \operatorname{tsr}(\mathcal{M}^{(n^+)}_+)_\mathbb{R}$.

Now we have proved our main theorem.

Proof of Theorem 1.2: The bsr and tsr results on $C(\mathbb{T}^n)$, $\ell^1(\mathbb{Z}^n)$, and $\mathcal{M}^{(n)}$ are from Corollary 5.14, those on $A(\mathbb{D}^n)$, $\ell^1(\mathbb{N}^n)$, and $\mathcal{M}^{(n^+)}_+$ are from Corollary 5.17, those on $\mathbb{C}\delta_0 + \mathrm{L}^1$ from Corollary 5.21, and those on $\mathbb{C}\delta_0 + \mathrm{L}^1_+$ from Corollary 5.22.

The results on $\mathcal{M}^{(n)} + \mathrm{L}^1$ and $\mathcal{M}^{(n^+)}_+ + \mathrm{L}^1_+$ are from Corollary 7.9. All "∞" entries are from Corollaries 6.3 and 6.4 (with Proposition 3.2). □

As one observes from the proofs, many of our results on $\mathcal{M}$ also hold for AP and for the classes in between. Indeed, in many results one can replace $\mathcal{M}$ by A and $\mathcal{M}_+$ by $A \cap \mathrm{AP}_+$ if $\mathcal{M} \subset A \subset \mathrm{AP}$, and A is a Banach algebra and a subalgebra of AP, and the sup-norm is continuous on A.

8. Discrete measures whose supports lie on $S \subset \mathbb{R}$

Let S be an additive subgroup of $\mathbb{R}$ (that is, $0 \in S = S - S \subset \mathbb{R}$). Then $\ell^1(S)$ and $\ell^1(S \cap \mathbb{R}_+)$ are closed subalgebras of $\ell^1(\mathbb{R})$. They are obviously isometrically isomorphic to $\mathcal{M}^{(S)}$ and $\mathcal{M}^{S \cap \mathbb{R}_+}$ (through $\ell^1(S) \ni a \mapsto \sum_{s \in S} a_s \delta_s \in \mathcal{M}^{(S)}$), where

$$\mathcal{M}^{(S)} := \{\mu \in \mathcal{M} \colon |\mu|(\mathbb{R} \setminus S) = 0\}, \tag{16}$$

the complex Banach algebra of discrete measures on S. If $\mu \in \mathcal{M}^n$, then we set $\mu_S := \sum_{r\in S} \mu(\{r\})\delta_r \in \mathcal{M}_S^n$ (note that $\mu = \sum_{r\in\mathbb{R}} \mu(\{r\})\delta_r$). We also set $\mathcal{M}_+^{(S)} := \mathcal{M}^{(S)} \cap \mathcal{M}_+ = \mathcal{M}^{S\cap\mathbb{R}_+}$. The algebra $\mathcal{M}^{(S)}$ was denoted by APW_S in [24].

In (the corona theorem) Proposition 8.2 we show that $i\mathbb{R}$ (respectively, $\mathbb{C}_+$) is dense in the maximal ideal space of $\mathcal{M}^{(S)}$ (respectively, $\mathcal{M}_+^{(S)}$). In Theorem 8.1 we compute the stable ranks of these algebras. Analogous results for the algebras AP^S and AP_+^S, the closures of $\mathcal{FM}^{(S)}$ and $\mathcal{FM}_+^{(S)}$ under the supremum norm, are given in Corollary 8.3. As a special case we obtain corresponding results for $\mathcal{M}_+^{(n)} := \mathcal{M}_n \cap \mathcal{M}_+$, which is isomorphic to $\ell^1(E)$, where $E := \{\alpha \in \mathbb{Z}^n : \alpha \cdot T \geq 0\}$.

But first we present the main result of this section. Here we show, for example, that $\operatorname{bsr} \mathcal{M}^{(S)} = \operatorname{bsr} \mathcal{M}_n = \lfloor n/2 \rfloor + 1$, where $n := \dim_{\mathbb{Q}} S$ (the $\mathbb{Q}$-dimension of the $\mathbb{Q}$-vector space S, possibly infinite).

Theorem 8.1. *Let S be an additive subgroup of $\mathbb{R}$. Let $n := \dim_{\mathbb{Q}} S$. Then*

$$\lfloor n/2 \rfloor + 1 = \operatorname{bsr} \mathcal{M}^{(S)} = \operatorname{bsr}(\mathcal{M}^{(S)})_{\mathbb{R}} = \operatorname{tsr} \mathcal{M}^{(S)} = \operatorname{tsr}(\mathcal{M}^{(S)})_{\mathbb{R}},$$

and $\lfloor n/2 \rfloor + 1 \leq \operatorname{bsr} \mathcal{M}_+^{(S)} \leq \operatorname{tsr} \mathcal{M}_+^{(S)} \leq \operatorname{tsr} \mathcal{M}_+^{(n)} \leq \lfloor (n+1)/2 \rfloor + 1$. In particular, if $n = 2k \geq 2$, $k \in \mathbb{N}$, then $\operatorname{bsr} \mathcal{M}_+^{(S)} = \operatorname{tsr} \mathcal{M}_+^{(S)} = k+1 = n/2+1$. Moreover, $2 = \operatorname{tsr} \mathcal{M}_+^{(S)} = \operatorname{bsr}(\mathcal{M}_+^{(S)})_{\mathbb{R}}$ if $n = 1$. The above also holds with $(\mathcal{M}_+^{(S)})_{\mathbb{R}}$ in place of $\mathcal{M}_+^{(S)}$.

The proof is given at the end of this section. Note that n does not characterize S; for example, we have $\dim_{\mathbb{Q}} S = 1$ for S equal to $\mathbb{Q}$, $\pi\mathbb{Z}$ or $\operatorname{span}_{\mathbb{Z}}\{7^{-k}3^{-j} : k, j \in \mathbb{N}\}$.

Recall from Theorems 7.7 and 7.6 that $\operatorname{bsr}(\mathcal{M}^{(S)} + \mathrm{L}^1) = \operatorname{bsr} \mathcal{M}^{(S)}$, $\operatorname{tsr}(\mathcal{M}^{(S)} + \mathrm{L}^1) = \operatorname{tsr} \mathcal{M}^{(S)}$, $\operatorname{bsr}(\mathcal{M}_+^{(S)} + \mathrm{L}_+^1) = \operatorname{bsr} \mathcal{M}_+^{(S)}$ etc.

Next we establish the corona theorem for $\mathcal{M}^{(S)}$, $\mathcal{M}_+^{(S)}$, $(\mathcal{M}^{(S)})_{\mathbb{R}}$ and $(\mathcal{M}_+^{(S)})_{\mathbb{R}}$.

Proposition 8.2. *Let $\mu \in (\mathcal{M}^{(S)})^n$, where S is as above. We have $\mu \in \mathrm{U}_n(\mathcal{M})$ iff $\mu \in \mathrm{U}_n(\mathcal{M}^{(S)})$.*

In fact, if $\nu \in \mathcal{M}^n$ and $\nu \cdot \mu = \delta_0$, then $\nu_S \cdot \mu = \delta_0$ (even $(\operatorname{Re} \nu_S) \cdot \mu = \delta_0$ if μ is real-valued; note that $\operatorname{Re} \nu_S \in \mathcal{M}^{(S)}$).

Thus, $\mu \in \mathrm{U}_n(\mathcal{M}_+) \Leftrightarrow \mu \in \mathrm{U}_n(\mathcal{M}_+^{(S)})$, and if μ is real-valued, then the left inverse can additionally be taken real-valued.

Proof. The first claim follows from the second, which we prove below. Let $s \in S$, $r \in \mathbb{R}$. Then $r \in S \Leftrightarrow r + s \in S$, and so for any $\mu \in \mathcal{M}^{(S)} \setminus \{0\}$ we have $r \notin S \Leftrightarrow (\delta_r * \mu)_S = 0$. Since $\delta_0 \in \mathcal{M}^{(S)}$, we conclude that if $\mu, \nu \in \mathcal{M}^n$ and $\nu \cdot \mu = \delta_0$, then $\nu_S \cdot \mu = \delta_0$. If $\mu = \operatorname{Re} \mu$, then trivially $(\operatorname{Re} \nu_S \cdot \mu) = \delta_0$.

The real claims follow by symmetrization (Lemma 4.4). □

The corona theorem for $\mathcal{M}_+^{(S)}$ was established already in [24, Theorem 2.4], with an upper bound for the left inverse.

From Proposition 8.2, we have $\mathrm{U}_n(\mathcal{M}^{(S)}) = \mathcal{M}_S^n \cap \mathrm{U}_n(\mathcal{M})$, so $\mathcal{M}^{(S)}$ is n-full in $\mathcal{M}$, for any $n \geq 1$. Therefore, $i\mathbb{R}$ (respectively, $\mathbb{C}_+$) is dense also in the maximal ideal space of $\mathcal{M}^{(S)} + \mathrm{L}^1$ (respectively, $\mathcal{M}_+^{(S)} + \mathrm{L}_+^1$), by Corollary 7.4.

Now we can establish the corona theorems and compute the stable ranks of corresponding algebras of almost periodic functions. Denote by AP^S and AP_+^S the closures of $\mathcal{FM}^{(S)}$ and

$\mathcal{FM}_+^{(S)}$, respectively, under the supremum norm. Thus, for example, AP_+^S is the closure in H^∞ of the linear combinations of the functions $\mathrm{e}^{-\cdot t}$, $t \in S \cap \mathbb{R}_+$.

Corollary 8.3 (AP^S)**.** *The set $i\mathbb{R}$ (respectively, $\mathbb{C}_+$) is dense in the maximal ideal space of AP^S (respectively, of AP_+^S).*

Thus, the corona condition $\inf_{i\mathbb{R}} |f| > 0$ (respectively, $\inf_{\mathbb{C}_+} |f| > 0$) is necessary and sufficient for $f \in \mathrm{U}_n(\mathcal{A})$, when $f \in \mathcal{A}^n$, where $\mathcal{A}$ equals AP^S or $(\mathrm{AP}^S)_\mathbb{R}$ (respectively, AP_+^S or $(\mathrm{AP}_+^S)_\mathbb{R}$).

Theorem 8.1 also holds with AP^S (respectively, AP_+^S) in place of $\mathcal{M}^{(S)}$ (respectively, $\mathcal{M}_+^{(S)}$).

Proof of Corollary 8.3:

1° The density of $\mathbb{C}_+$ in $X(\mathrm{AP}_+^S)$ is [24, Theorem 2.3] (based on [1]).

2° Assume that $f \in (\mathrm{AP}^S)^n$ and $\epsilon := \inf_{i\mathbb{R}} |f| > 0$. Let $\mathcal{FM}^{(S)^n} \ni g_k \to f$, as $k \to \infty$. If $n = 1$, then $g_k^{-1} \to f^{-1}$ in AP. Hence $f^{-1} \in \mathrm{AP}^S$, that is, $f \in \mathrm{U}_n(\mathrm{AP}^S)$.

3° For general n, we have $|f|^2 = \bar{f} \cdot f \in \mathrm{AP}^S$ (because $\overline{\mathrm{e}^{irt}} = \mathrm{e}^{ir(-t)}$, and $-t \in S$ if $t \in S$). Thus $|f|^{-2} \in \mathrm{AP}^S$, by 3°, and so $|f|^{-2}\bar{f} \in (\mathrm{AP}^S)^n$. As $|f|^{-2}\bar{f} \cdot f = 1$, it follows that $f \in \mathrm{U}_n(\mathrm{AP}^S)$ in this case as well.

4° From Lemma 4.4 we see that the corona conditions also apply to $(\mathrm{AP}^S)_\mathbb{R}$ and to $(\mathrm{AP}_+^S)_\mathbb{R}$.

5° By Corollary 3.8 (with $f := \mathcal{F}$; fullness from 4°), it follows that the bounds of the bsr and tsr of $\mathcal{M}^{(S)}$ (respectively, $\mathcal{M}_+^{(S)}$, $(\mathcal{M}^{(S)})_\mathbb{R}$, $(\mathcal{M}_+^{(S)})_\mathbb{R}$) in Theorem 8.1 also apply to AP^S (respectively, AP_+^S, $(\mathrm{AP}^S)_\mathbb{R}$, $(\mathrm{AP}_+^S)_\mathbb{R}$). (To get $\operatorname{tsr} \mathrm{AP}_+^S = 2$ in the case $n = 1$, use the proof of Theorem 8.1.)

□

Next we study an important special case of the algebra $\mathcal{M}_+^{(S)}$, namely $\mathcal{M}_+^{(n)} := \mathcal{M}^{(n)} \cap \mathcal{M}_+$. Corresponding results are then used to prove Theorem 8.1.

Recall that $T_1, T_2, \dots, T_n > 0$ are $\mathbb{Q}$-independent. Obviously, $\mathcal{M}_+^{(n^+)} \subset \mathcal{M}_+^{(n)} := \mathcal{M}^{(n)} \cap \mathcal{M}_+$, and the inclusion is strict iff $n \geq 2$ (for example, $\delta_{T_2} - \delta_{T_1} \in \mathcal{M}_+^{(n)} \setminus \mathcal{M}_+^{(n^+)}$). If $T := (T_1, \dots, T_n)$ and $E := \{\alpha \in \mathbb{Z}^n \colon \alpha \cdot T \geq 0\}$, then

$$\ell^1(E) \ni a \mapsto \sum_{\alpha \in E} a_\alpha \delta_T^\alpha, = \sum_{\alpha \in E} a_\alpha \delta_{\alpha \cdot T} \tag{17}$$

is an isometric isomorphism of $\ell^1(E)$ onto $\mathcal{M}_+^{(n)}$. Its restriction to $\ell^1(\mathbb{N}^n)$ is the isomorphism (4) of $\ell^1(\mathbb{N}^n)$ onto $\mathcal{M}_+^{(n^+)}$. If $S := \mathbb{Z}T_1 + \mathbb{Z}T_2 + \cdots + \mathbb{Z}T_n \subset \mathbb{R}$, then $\mathcal{M}_+^{(n)} = \mathcal{M}_+^{(S)}$. Hence $\mathcal{M}_+^{(n)}$ is also isometrically isomorphic to $\ell^1(S)$. By Proposition 8.2, it follows that $\mathbb{C}_+$ is dense in $X(\mathcal{M}_+^{(n)})$; recall that the same is not true for $X(\mathcal{M}_+^{(n^+)})$ (for $n \geq 2$). Note also that $\mathcal{FM}_+^{(n)}$ is the closure in $\mathcal{FM}_+$ of the algebra generated by the functions $\mathrm{e}^{-\cdot t}$, where $0 \leq t = m_1 T_1 + \cdots + m_n T_n$ for some $m_1, \dots, m_n \in \mathbb{Z}$.

Lemma 8.4 ($X(\mathcal{M}_+^{(n)})$)**.** *The maximal ideal space of $\ell^1(E)$ (or of $\mathcal{M}_+^{(n)}$) equals*

$$X_n := \{z \in \overline{\mathbb{D}}^n \colon |z_k|^{1/T_k} = |z_1|^{1/T_1} \quad \forall k \leq n\}$$

through the Gelfand transform $\ell^1(E) \ni a \mapsto \widehat{a} \in C(X_n)$, where $\widehat{a}(z) := \sum_{\alpha \in E} a_\alpha z^\alpha$.

Here $\widehat{a}(0) := a_{(0,\dots,0)}$; this obviously makes $\widehat{a}$ continuous on X_n.

Proof. It is easy to see that $\widehat{(ab)} = \widehat{a}\widehat{b}$, and so X_n is contained in the maximal ideal space. Its topology is stronger than that inherited from $\overline{\mathbb{D}}^n$ (which is the maximal ideal space of $\ell^1(\mathbb{N}^n)$), since $\widehat{a}$ must be continuous for every $a \in \ell^1(\mathbb{N}^n)$. On the other hand, for every $a \in \ell^1(E)$, the transform $\widehat{a}$ is continuous on X_n in the inherited topology, and so the two coincide. But X_n is closed, hence we only need to show that it is dense in the maximal ideal space.

Set $\phi(s) := \mathrm{e}^{-Ts}$. Then $\phi[\overline{\mathbb{C}_+}]$ is a dense subset of X_n (in the same way as $\phi[i\mathbb{R}]$ is a dense subset of $\mathbb{T}^n$, by Kronecker's Theorem). Let $k \geq 1$ and $a \in \ell^1(E)^k$. With $f := \widehat{a} \circ \phi$, we have $f(s) = \sum_{\alpha \in E} a_\alpha \mathrm{e}^{-sT\cdot\alpha}$, and so $f \in \mathrm{U}_k(\mathcal{F}\mathcal{M}_+^{(n)})$ iff $\inf_{\mathbb{C}_+} |f| > 0$, by [24, Theorem 2.3], or equivalently, iff $\inf_{X_n} |\widehat{a}| > 0$. By Lemma 4.2, it follows that X_n is dense in the maximal ideal space $X(\mathcal{M}_+^{(n)})$. □

Using the above result on the maximal ideal space, we can now compute the stable ranks of these algebras. Note that this is a special case of Theorem 8.1.

Lemma 8.5 ($\mathrm{bsr}\, \mathcal{M}_+^{(n)}$)**.** *Let $k, n \in \mathbb{N}$. Let $\mathcal{A}$ denote $\mathcal{M}_+^{(n)}$ or $(\mathcal{M}_+^{(n)})_\mathbb{R}$.*

(1) *If $n = 2k \geq 2$, then* $\mathrm{bsr}\, \mathcal{A} = \mathrm{tsr}\, \mathcal{A} = k + 1 = n/2 + 1$.

(2) *If $n = 2k + 1$, then $\lfloor n/2 \rfloor + 1 = k + 1 \leq \mathrm{bsr}\, \mathcal{A} \leq \mathrm{tsr}\, \mathcal{A} \leq k + 2 = \lfloor (n + 1)/2 \rfloor + 1$.*

Proof. Let $\mathcal{A} = \mathcal{F}\mathcal{M}_+^{(n)}$ or $\mathcal{A} = \mathcal{F}(\mathcal{M}_+^{(n)})_\mathbb{R}$. By Corollary 6.3 we have $\lfloor n/2 \rfloor + 1 \leq \mathrm{bsr}\, \mathcal{A}$, so we only have to show that $\lfloor (n + 1)/2 \rfloor + 1 \geq \mathrm{tsr}\, \mathcal{A}$.

If $k := \lfloor (n+1)/2 \rfloor + 1$, then $2k > n+1$. Let $f \in \mathcal{A}^k$ and $\epsilon > 0$ be given. We shall construct $G \in \mathrm{U}_k(\mathcal{A})$ such that $\|f - G\|_\mathcal{A} < \epsilon$.

Pick $a \in \ell^1(E)^k$ corresponding to $\mathcal{F}^{-1}f$ (see (17)). By density, we can assume that a has a finite support. It follows that $\widehat{a}$ is holomorphic $\mathbb{C}_*^n \to \mathbb{C}^k$, where $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$.

Set $b_j(z) := \mathrm{Im}\, \widehat{a}_j(z)/\, \mathrm{Im}\, z_j$, $F := (\mathrm{Re}\, \widehat{a}, b) \in C(V; \mathbb{R}^{2k})$, where

$$V := \{z \in \mathbb{C}^n \colon |z_j|^{1/T_j} = |z_1|^{1/T_1} \quad \forall j \leq n\}.$$

Since V is $n + 1$-dimensional, by Lemma A.2 $W := \mathbb{R}^{2k} \setminus F[X_n \setminus \{0\}]$ is dense. Indeed, define $\phi \in C^\infty(\mathbb{R}^n \times (0, \infty); \mathbb{R}^{2k})$ by $\phi := (x_{n+1}^{T_1/T_1} \mathrm{e}^{ix_1}, x_{n+1}^{T_2/T_1} \mathrm{e}^{ix_2}, \ldots, x_{n+1}^{T_n/T_1} \mathrm{e}^{ix_n})$, and set $A := [0, 2\pi)^n \times (0, 1]$. Then $\phi(A) = X_n \setminus \{0\}$, and $F \circ \phi$ is differentiable (even C^∞). Hence $F[X_n \setminus \{0\}] = (F \circ \phi)[A]$ has measure zero, by Lemma A.2, and so W is dense. Pick $r, t \in \mathbb{R}^k \setminus \{\widehat{a}(0)\}$ such that $(r; t) \in W$. If $\widehat{c}(z) := \widehat{a}(z) - r - (t_1 z_1, \ldots, t_n z_n) \quad z \in X_n$, then $\widehat{c}(0) = \widehat{a}(0) - r \neq 0$, and for $z \in X_n \setminus \{0\}$ we have $(\mathrm{Re}\, \widehat{c}_j(z), \mathrm{Im}\, \widehat{c}_j(z)/\, \mathrm{Im}\, z_j) = (\mathrm{Re}\, \widehat{a}(z)_j - r_j, b_j(z) - t_j)$, so $\widehat{c}_j(z) = 0\ \forall j \leq k$ would imply that $F(z) = (r; t) \in W$, a contradiction. Thus $\widehat{c}$ has no zeros on X_n, and so $c \in \mathrm{U}_k(\ell^1(E))$, from Lemma 8.4. Since we can have (r, t) arbitrarily small, we can have $\|a - c\|_{\ell^1(E)} < \epsilon$, that is, $\|f - g\|_{\mathcal{A}^k} < \epsilon$, where $g \in \mathrm{U}_k(\mathcal{A})$ equals $\sum_\alpha c_\alpha \mathrm{e}^{-\cdot \sum_{l=1}^n T_l \alpha_l}$ (that is, g is the Laplace transform of the right-hand-side of (17) with c in place of a). □

We formulate here the remark relevant to Mortini's question (see Remark 6.5).

Remark 8.6 (AP_+^n)**.** Denote by $\mathrm{AP}_+^n \subset \mathrm{AP}_+$ the closure of $\mathcal{F}\mathcal{M}_+^{(n)}$ in AP. It obviously equals AP^S, the closure in $H^\infty(\mathbb{C}_+)$ of the algebra generated by the functions $\mathrm{e}^{-\cdot t}$, where $0 \leq t = m_1 T_1 + \cdots + m_n T_n$ for some $m_1, \ldots, m_n \in \mathbb{Z}$. Here $S := \mathbb{Z}T_1 + \cdots + \mathbb{Z}T_1$.

By Corollary 8.3, AP_+^n is full in $H^\infty(\mathbb{C}_+)$ and $\mathrm{bsr}\, \mathrm{AP}_+^{2k} = k + 1$ for $k = 1, 2, \ldots$.

Proof of Theorem 8.1: The case $n = 0$ is trivial, so assume that $n \geq 1$.

1° We show that $\lfloor n/2 \rfloor + 1 \leq \mathrm{bsr}\, \mathcal{A}$. Since $\mathbb{Q}$-dim $S = n$, and $S = S - S \subset \mathbb{R}$, there exist $\mathbb{Q}$-independent $T_1, T_2, \ldots, T_n \subset S \cap \mathbb{R}_+$. Let S' be the set of (finite) $\mathbb{Z}$-linear combinations of $T_1, \ldots, T_n$. Since S is a subgroup of $\mathbb{R}$, we have $S' \subset S$, hence $\mathcal{M}^{(n)} = \mathcal{M}^{S'} \subset \mathcal{M}^{(S)}$; similarly, $(\mathcal{M}^{(n)})_\mathbb{R} \subset (\mathcal{M}^{(S)})_\mathbb{R}$, $\mathcal{M}_+^{(n^+)} \subset \mathcal{M}_+^{(S)}$, and $(\mathcal{M}_+^{(n^+)})_\mathbb{R} \subset (\mathcal{M}_+^{(S)})_\mathbb{R}$.

By Corollary 6.3, this implies that $\lfloor n/2 \rfloor + 1 \leq \operatorname{bsr} \mathcal{A}$, where $\mathcal{A}$ equals $(\mathcal{M}_+^{(S)})_\mathbb{R}$, $\mathcal{M}_+^{(S)}$, $(\mathcal{M}^{(S)})_\mathbb{R}$, or $\mathcal{M}^{(S)}$ (since $(\mathcal{M}_+^{(S)})_\mathbb{R} \subset \mathcal{A}$ and $\mathcal{F}\mathcal{A} \subset \mathcal{M} \subset \mathrm{AP}$).

(If $n = \infty$, then apply the above for every finite n to observe that $\operatorname{bsr} \mathcal{A} = \infty$ and hence $\operatorname{tsr} \mathcal{A} = \infty$. Thus, we may assume that $n < \infty$.)

2° We have $\operatorname{tsr} \mathcal{M}_+^{(S)} \leq \operatorname{tsr} \mathcal{M}_+^{(n)} =: m$. Let $f \in (\mathcal{M}_+^{(S)})^m$, $\epsilon > 0$. By density, we can assume that the support A of f is finite. Define $T_1, \ldots, T_n$ by Lemma 6.1 (if $l := \mathbb{Q}\text{-}\dim A < n$, pick $T_{l+1}, \ldots, T_n > 0$ so as to get a $\mathbb{Q}$-basis of S). Since $A \subset \mathbb{Z}T_1 + \ldots + \mathbb{Z}T_n \subset S$ and $A \subset \mathbb{R}_+$, we have $f \in (\mathcal{M}_+^{(n)})^m$ and $\mathcal{M}_+^{(n)} \subset \mathcal{M}_+^{(S)}$.

Since $\operatorname{tsr} \mathcal{M}_+^{(n)} = m$, there exists $g \in \mathrm{U}_m(\mathcal{M}_+^{(n)})$ such that $\|f - g\|_\mathcal{M} < \epsilon$. But $\mathrm{U}_m(\mathcal{M}_+^{(n)}) \subset \mathrm{U}_m(\mathcal{M}_+^{(S)})$, because $\mathcal{M}_+^{(n)} \subset \mathcal{M}_+^{(S)}$, and so $g \in \mathrm{U}_m(\mathcal{M}_+^{(S)})$. As f and ϵ were arbitrary, $\operatorname{tsr} \mathcal{M}_+^{(S)} \leq m$.

3° As in 2°, we see that $\operatorname{tsr}(\mathcal{M}_+^{(S)})_\mathbb{R} \leq \operatorname{tsr}(\mathcal{M}_+^{(n)})_\mathbb{R}$, $\operatorname{tsr} \mathcal{M}^{(S)} \leq \operatorname{tsr} \mathcal{M}^{(n)}$, and $\operatorname{tsr}(\mathcal{M}^{(S)})_\mathbb{R} \leq \operatorname{tsr}(\mathcal{M}^{(n)})_\mathbb{R}$. This and the known tsr results for the latter algebras (Corollary 5.14 and Lemma 8.5) and the fact that $\operatorname{bsr} \mathcal{A} \leq \operatorname{tsr} \mathcal{A}$ (Proposition 3.2) yield Theorem 8.1 except for the claims on the case $n = 1$.

4° Assume that $n = 1$. Let $0 < t \in S$. If $\mu := \delta_t - \mathrm{e}^{-1}\delta_0$, then $\widehat{\mu}(s) = \mathrm{e}^{-ts} - \mathrm{e}^{-1}$, hence $\widehat{\mu}(1/t) = 0$. By Lemma 5.11, $\operatorname{tsr} \mathcal{M}_+^{(S)} \geq 2$, because $\mu \in \mathcal{M}_+^{(S)}$. By the above, $\operatorname{tsr} \mathcal{M}_+^{(S)} \leq \lfloor (2+1)/2 \rfloor + 1 = 2$, and so $\operatorname{tsr} \mathcal{M}_+^{(S)} = 2$. Similarly, $\operatorname{tsr}(\mathcal{M}_+^{(S)})_\mathbb{R} = 2$.

Finally, with $g(s) := (\mathrm{e}^{-ts} - 1)(\mathrm{e}^{-ts} - 1/3)$, $f(s) := \mathrm{e}^{-ts} - 1/2$, we have that $(f, g) \in \mathcal{F}\mathrm{U}_2((\mathcal{M}_+^{(S)})_\mathbb{R})$. Given $h \in \mathcal{F}(\mathcal{M}_+^{(S)})_\mathbb{R}$, the function f (hence also $f + hg$) has different signs at the zeros of g, and so by the mean-value theorem, $f + hg$, being real-valued, has a zero on $\mathbb{R}_+$. Consequently (f, g) is not reducible, and so $\operatorname{bsr}(\mathcal{M}_+^{(S)})_\mathbb{R} \geq 2$. Thus $\operatorname{bsr}(\mathcal{M}_+^{(S)})_\mathbb{R} = 2$.

□

9. Exponentially stable subalgebras

In this section we define and study exponentially stable measures and functions and exponentially (actually "power") stable ℓ^1 sequences. These classes were introduced below Theorem 1.2. The impulse responses or transfer functions of "exponentially stable" continuous-time systems or of "power stable" discrete-time systems are often of one of these forms. This is why such classes are often studied in the literature (including the *Callier–Desoer class* of fractions of elements of $(\mathcal{M}_+ + \mathrm{L}^1(\mathbb{R}_+))^{\exp}$ [4]).

The main result of this section is Theorem 9.4, which states that all results of Table 1 hold also with $\mathcal{A}$ (or $\mathcal{A}_\mathbb{R}$) replaced by the corresponding exponential subalgebra. Analogous results on $\mathcal{M}^{(S)}$, $\mathcal{M}_+^{(S)}$, AP^S, $\mathcal{M}_+^{(n)}$ etc. are given in Theorem 9.6. We also study unimodularity and other properties in all these algebras. Actually, all results of §7 and §8 hold for the exponential case too.

The algebra $\mathcal{F}\mathcal{M}_+^{\exp}$ consists of functions $\widehat{f}(\omega + \cdot)$, where $\widehat{f} \in \mathcal{F}\mathcal{M}_+$ and $\omega > 0$. We generalize this as follows.

Definition 9.1 ($\mathcal{A}^{\exp}$)**.** We write $f \in (\mathcal{M}+\mathrm{L}^1)^{\exp}$ if $\mathrm{e}^{\omega\cdot} f \in \mathcal{M}+\mathrm{L}^1$ for some $\omega > 0$ and for some $\omega < 0$. If $\mathcal{A}$ is a subalgebra of $\mathcal{M}+\mathrm{L}^1$, then we set

$$\mathcal{A}^{\exp} := \mathcal{A} \cap (\mathcal{M}+\mathrm{L}^1)^{\exp}, \quad \mathcal{A}_{\mathbb{R}}^{\exp} := \mathcal{A}_{\mathbb{R}} \cap (\mathcal{M}+\mathrm{L}^1)^{\exp}.$$

We write $a \in \ell^{1,\exp}(\mathbb{R})$ if $\sum_{r\in\mathbb{R}} a_r\delta_r \in \mathcal{M}^{\exp}$, or equivalently, if $(q^r a_r)_{r\in\mathbb{R}} \in \ell^1(\mathbb{R})$ for some $q > 1$ and for some $q \in (0,1)$; similarly for the complex and real subclasses of $\ell^1(\mathbb{R})$. We write $a \in \ell^{1,\exp}(\mathbb{Z}^n)$ if $\sum_{\alpha\in\mathbb{Z}^n} a_\alpha \delta_{\alpha\cdot T} \in \mathcal{M}^{\exp}$ (equivalently, $\in \mathcal{M}^{(n),\exp}$); similarly for $\ell^{1,\exp}(\mathbb{N}^n)$ and their real subclasses. We use the same norms as in the original algebras.

In the literature, $\mathcal{A}^{\exp}$ is sometimes denoted by $\mathcal{A}_-$. All these *exponential algebras* are obviously subalgebras of the original ones. Observe that $f \in (\mathbb{C}\delta_0+\mathrm{L}^1)_+^{\exp}$ iff $\mathrm{e}^{\omega\cdot} f \in \mathbb{C}\delta_0 + \mathrm{L}^1_+$ for some $\omega > 0$, that is, iff $f = f_\mathrm{a} + c\delta_0$, where $c \in \mathbb{C}$ and $\mathrm{e}^{\omega\cdot} f \in \mathrm{L}^1(\mathbb{R}_+)$ for some $\omega > 0$. An analogous claim holds for any of the other causal (support on $\mathbb{R}_+$ or on $\mathbb{N}$) algebras.

Remark 9.2. Remark 2.1 holds for these exponential classes too; in particular the canonical isomorphism (4) of $\ell^1(\mathbb{N}^n)$ onto $\mathcal{M}_+^{(n^+)}$ maps $\ell^{1,\exp}(\mathbb{N}^n)$ onto $\mathcal{M}_+^{(n^+),\exp}$. Analogously, $\ell^{1,\exp}(\mathbb{Z}^n) \approx \mathcal{M}^{(n),\exp}$.

We have $a \in \ell^{1,\exp}(\mathbb{R})$ iff $(q^r a_r)_{r\in\mathbb{R}} \in \ell^1(\mathbb{R})$ for some $q > 1$ and for some $q \in (0,1)$. Moreover, $\ell^{1,\exp}(\mathbb{Z}) = \ell^{1,\exp}(\mathbb{R}) \cap \ell^1(\mathbb{Z})$, $\ell^{1,\exp}(\mathbb{Z};\mathbb{R}) = \ell^{1,\exp}(\mathbb{Z}) \cup \ell^1(\mathbb{Z};\mathbb{R})$, etc. Furthermore, $a \in \ell^{1,\exp}(\mathbb{N}^n)$ iff if $(q^{|\alpha|}a_\alpha)_{\alpha\in\mathbb{N}^n} \in \ell^1(\mathbb{N}^n)$ for some $q > 1$.

Thus, $\ell^{1,\exp}(\mathbb{N}^n)$ is independent of $T := (T_1, T_2, \ldots, T_n)$; the same is not true for $\ell^{1,\exp}(\mathbb{Z}^n)$.

Proof. The first paragraph is a direct consequence of Definition 9.1. For $f = \sum_{r\in\mathbb{R}} a_r\delta_r$ the condition $\mathrm{e}^{\omega\cdot} f \in \mathcal{M}$ means that $\sum_{r\in\mathbb{R}} \mathrm{e}^{\omega r}|a_r| < \infty$, that is, that $\sum_{r\in\mathbb{R}} q^r|a_r| < \infty$, where $q := \mathrm{e}^\omega$. Obviously, we can have $q > 1$ (respectively, $q < 1$) iff we can have $\omega > 0$ (respectively, $\omega < 0$). The claims on $\ell^{1,\exp}(\mathbb{Z})$ and $\ell^{1,\exp}(\mathbb{Z};\mathbb{R})$ obviously follow.

If $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha \delta_{\alpha\cdot T}$, then

$$\|q_1^{|\cdot|}a.\|_{\ell^1(\mathbb{N}^n)} = \sum_{\alpha\in\mathbb{N}^n} |a_\alpha| \mathrm{e}^{\omega T_1 \sum_{k=1}^n \alpha_k} \le \sum_{\alpha\in\mathbb{N}^n} |a_\alpha| \mathrm{e}^{\omega(\alpha\cdot T)} \le \|q_n^{|\cdot|}a.\|_{\ell^1(\mathbb{N}^n)} \tag{18}$$

when $\omega > 0$, where $q_k := \mathrm{e}^{\omega T_k}$, $|\alpha| := \sum_{k=1}^n \alpha_k$. Thus, (18) is finite for some $\omega > 0$ iff $\|q^{|\cdot|}a.\|_{\ell^1(\mathbb{N}^n)} < \infty$ for some $q > 1$. Equivalently, $f \in \mathcal{M}^{\exp}$ (hence $f \in \mathcal{M}_+^{(n^+),\exp}$) iff $a \in \ell^{1,\exp}(\mathbb{N}^n)$. □

By $\mathrm{A}^{\exp}(\mathbb{D}^n) = H^{\infty,\exp}(\mathbb{D}^n)$ (respectively, $\mathrm{C}^{\exp}(\mathbb{T}^n)$) we denote the algebra of functions that are holomorphic on a neighborhood of $\overline{\mathbb{D}}^n$ (respectively, of $\mathbb{T}^n$). Let $S \subset \mathbb{R}$. By $\mathrm{AP}_S^{\exp}$ we denote the functions that are uniform limits of linear combinations of $\mathrm{e}^{-r\cdot}$ ($r \in S$) on $\{-\omega \le \operatorname{Re} z \le \omega\}$ for some $\omega > 0$. We set $(\mathrm{AP}_S^+)^{\exp} := \mathrm{AP}^+_{S\cap\mathbb{R}_+}$, $\mathrm{AP}^{\exp} := \mathrm{AP}_{\mathbb{R}}^{\exp}$, $\mathrm{AP}_+^{\exp} := \mathrm{AP}_{\mathbb{R}_+}^{\exp}$. Obviously, $\mathrm{AP}_+^{\exp}$ consists of all functions of the form $f(\cdot+\omega)$ ($f \in \mathrm{AP}_+$, $\omega > 0$). Moreover, $\mathcal{F}\mathcal{M}^{\exp} \subset \mathrm{AP}^{\exp}$ and $\mathrm{AP}_+^{\exp} = \mathrm{AP}_+ \cap \mathrm{AP}^{\exp}$.

Next we show that left-invertibility (that is, unimodularity) in an algebra is equivalent to left-invertibility in the corresponding exponential algebra. For example, if $f \in (\mathcal{M}^{\exp})^n$ and $g \cdot f = 1$ for some $g \in \mathcal{M}^n$, then $g \cdot f = 1$ for some $g \in (\mathcal{M}^{\exp})^n$.

Theorem 9.3 (Exponential Corona Theorem)**.** *Any exponential algebra (say, $\mathcal{A}^{\exp}$) defined above is dense and full in the corresponding original algebra (say, $\mathcal{A}$) except that $H^{\infty,\exp}$ is not dense in H^∞. Consequently,*

$$\mathrm{U}_n(\mathcal{A}^{\exp}) = (\mathcal{A}^{\exp})^n \cap \mathrm{U}_n(\mathcal{A}), \text{ and } \mathrm{U}_n(\mathcal{A}_{\mathbb{R}}^{\exp}) = (\mathcal{A}_{\mathbb{R}}^{\exp})^n \cap \mathrm{U}_n(\mathcal{A}). \tag{19}$$

In particular, Corona Theorems 4.3 and 4.5 hold even if we replace each algebra by the corresponding exponential algebra.

By "defined above" in Theorem 9.3 we mean that $\mathcal{A}^{\exp}$ has explicitly been defined above (in this section) or that it is the exponential algebra of a subalgebra of $\mathcal{M}+\mathrm{L}^1$ or of $(\mathcal{M}+\mathrm{L}^1)_{\mathbb{R}}$ that has been defined in the above sections. Thus, $\mathcal{A}$ can be $\mathcal{M}^{(n)}$, $\mathcal{M}_+^{(n^+)}$, $\mathcal{M}_+^{(n)}$, $\mathcal{M}^{(S)}$ or $\mathcal{M}_+^{(S)}$, or such an algebra $+\mathrm{L}^1$ or $+\mathrm{L}^1_+$ (depending on causality), or any of the algebras of §1–2, or the real algebra corresponding to any of these algebras. However, Theorem 9.3 holds also for those typical exponential subalgebras that were not defined above.

We note that for $\mathcal{A} = \mathcal{M}_+ + \mathrm{L}^1_+$ and $n = 2$ the corona theorem for $\mathcal{A}^{\exp}$ was established in [4].

If $\mathcal{A}$ is an n-full subalgebra of some other algebra $\mathcal{A}'$ (e.g., of $\mathcal{M} + \mathrm{L}^1$), then, obviously, $\mathrm{U}_n(\mathcal{A})$ can be replaced by $\mathrm{U}_n(\mathcal{A}')$ in (19).

Proof of Theorem 9.3: Density follows from Lemma A.1. Below we prove fullness, that is, that $\mathcal{A}^{\exp}\cap \mathrm{U}_1(\mathcal{A}) = \mathrm{U}_1(\mathcal{A}^{\exp})$ (only "$\subset$" needs to be proved). Then (19) follows from Lemma 3.5 (with $\mathcal{A}_{\mathbb{R}}$ in place of the latter $\mathcal{A}$, hence as is too), and (19) implies the modified corona theorems.

1° If $f \in \mathrm{A}^{\exp}(\mathbb{D}^n) \cap \mathrm{U}_1(A(\mathbb{D}^n))$, then $|f| > \epsilon > 0$ on $\overline{\mathbb{D}}$, hence on a neighborhood of $\overline{\mathbb{D}}$, hence then $f^{-1} \in \mathrm{A}^{\exp}(\mathbb{D}^n)$. Since $H^{\infty,\exp}(\mathbb{D}^n) = \mathrm{A}^{\exp}(\mathbb{D}^n)$, this proves the fullness of $H^{\infty,\exp}(\mathbb{D}^n)$ in $H^{\infty}(\mathbb{D}^n)$ too. The proof for $C(\mathbb{T}^n)$ is analogous.

2° Assume that $\mathcal{A} = \mathrm{AP}_S$. Let $p_n \to f$ uniformly on a strip $E_\omega := \{-\omega \le \operatorname{Re} z \le \omega\}$ as in the definition of $\mathrm{AP}_S^{\exp}$. If $f \in \mathrm{U}_1(\mathrm{AP}_S)$, then, by uniform continuity, there exist $r, \epsilon > 0$ such that $|f| > \epsilon$ on E_r. Consequently, $\mathcal{F}\mathcal{M}^{\exp} \ni p_n^{-1} \to f^{-1}$ uniformly on E_r, hence $f^{-1} \in \mathrm{AP}_S^{\exp}$.

3° Assume that $\mathcal{A}$ is a closed subalgebra of $\mathcal{M} + \mathrm{L}^1$ with the same norm. Let $f \in \mathcal{A}^{\exp}\cap\mathrm{U}_1(\mathcal{A})$. The map $\alpha \mapsto f_\alpha := \mathrm{e}^{-\alpha \cdot} f$ is continuous $(-\delta, \delta) \to \mathcal{A}$, by the Dominated Convergence Theorem, for some $\delta > 0$, and $f_\alpha \in \mathrm{U}_1(\mathcal{A})$ (when δ is small enough, by Lemma 3.4). For any such α, define $g_\alpha \in \mathcal{A}$ by $g_\alpha * f_\alpha = 1$. Obviously, $(g_\alpha)_{-\alpha} * (f_\alpha)_{-\alpha} = 1$, hence $(g_\alpha)_{-\alpha} = g_0$. Consequently, $g_0 \in \mathcal{A}^{\exp}$, so $f_\alpha \in \mathrm{U}_1(\mathcal{A}^{\exp})$.

4° For all $\ell^{1,\exp}$ algebras we get the results from Remarks 9.2 and 2.1.

□

Theorem 9.4 (bsr $\mathcal{A}^{\exp}$)**.** *All results in Table 1 hold with $\mathcal{A}^{\exp}$ in place of $\mathcal{A}$.*

Proof. By Corollary 3.8 and Theorem 9.3, we have $\operatorname{bsr}\mathcal{A}^{\exp} \le \operatorname{bsr}\mathcal{A} \le \operatorname{tsr}\mathcal{A} = \operatorname{tsr}\mathcal{A}^{\exp}$ in each case (also the real ones), so only $\operatorname{bsr}\mathcal{A} \le \operatorname{bsr}\mathcal{A}^{\exp}$ needs to be proved (although in all cases below save the last, $\operatorname{bsr}\mathcal{A} = \operatorname{bsr}\mathcal{A}^{\exp}$ is proved directly).

For $A(\mathbb{D}^n)_{\mathbb{R}}$ and $\mathcal{F}\ell^1(\mathbb{N}^n)$ this follows from Corollary 5.4. For $C(\mathbb{T}^n)$, $C(\mathbb{T}^n)_{\mathbb{R}}$, $\mathcal{F}\ell^1(\mathbb{Z}^n)$ and $\mathcal{F}\ell^1(\mathbb{Z}^n;\mathbb{R})$ use Corollary 5.15. For $A(\mathbb{D}^n)$, $A(\mathbb{D}^n)_{\mathbb{R}}$, $\mathcal{F}\ell^1(\mathbb{N}^n)$ and $\mathcal{F}\ell^1(\mathbb{N}^n;\mathbb{R})$ use Corollary 5.18.

From Lemma 5.20 we observe that Corollary 5.15 (respectively, 5.18) can be applied to (the exponential subalgebras of) $\mathbb{C}\delta_0 + \mathrm{L}^1$ and $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R};\mathbb{R})$ (respectively, $\mathbb{C}\delta_0 + \mathrm{L}^1_+$ and $\mathbb{R}\delta_0+\mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$). Lemma 9.7 covers, among others, the (exponential versions of the) algebras listed at the end of Corollary 6.3.

By Remark 9.2 we get the remaining ℓ^1 algebras. From Lemma 3.9 we get the remaining algebras. □

In particular, we have $\mathrm{bsr}(\mathcal{M}_+ + \mathrm{L}^1(\mathbb{R}_+))^{\exp} = \infty$, so not all Callier–Desoer transfer functions g/f can be reduced (strongly stabilized). However, when the pair $(f,g) \in \mathrm{U}_2$ happens to lie in a nice subalgebra of $\mathcal{F}(\mathcal{M}_+ + \mathrm{L}^1(\mathbb{R}_+))^{\exp}$, such as $\mathcal{F}(\mathcal{M}_+^{(1^+)} + \mathrm{L}^1(\mathbb{R}_+))^{\exp}$, then it can be reduced, since $\mathrm{bsr}(\mathcal{M}_+^{(1^+)} + \mathrm{L}^1(\mathbb{R}_+))^{\exp} = 1$, by Theorem 9.4.

Lemma 9.5. *All results in §7 hold even if we replace all algebras by their exponential subalgebras. Moreover, in Theorems 7.6 and 7.7, with the additional assumption that $\mathrm{U}_n(\mathcal{A}^{\exp})$ is open in $(\mathcal{A}^{\exp})^n$, it suffices to make the replacements only in the conclusions.*

Proof. This follows by making the corresponding changes in the proofs too (in the proof of Lemma 7.1 we need the result $\mathrm{bsr}((\mathbb{C}\delta_0 + \mathrm{L}^1)_+^{\exp}) = 1$; also some exponential corona results are used). The only exception is Corollary 7.9, which is established in Theorem 9.4 (using only the above part of this lemma). □

The analogy of Theorem 9.4 holds for the $\mathcal{M}^{(S)}$ classes too.

Theorem 9.6. *All results of §8 hold even if we replace all algebras by their exponential subalgebras.*

Thus, e.g., $\mathrm{bsr}\,\mathcal{M}_+^{(S),\exp} = \mathrm{tsr}\,\mathcal{M}_+^{(S),\exp} = k+1$ when $\dim_{\mathbb{Q}} S = 2k$.

Proof. The "maximal ideal space" (set of nonzero continuous homomorphisms) of a normed algebra obviously equals that of its closure, so the first claim of Corollary 8.3 remains.

Corollary 3.8 (using Theorem 9.3) implies that $\mathrm{bsr}\,\mathcal{A}^{\exp} \le \mathrm{bsr}\,\mathcal{A} \le \mathrm{tsr}\,\mathcal{A} = \mathrm{tsr}\,\mathcal{A}^{\exp}$ for all algebras treated in §8. But $\lfloor n/2 \rfloor + 1 \le \mathrm{bsr}\,\mathcal{A}^{\exp}$, by Lemma 9.7 and Corollary 6.3 (we have $\mathcal{M}_+^{(n^+),\exp} \subset \mathcal{A}^{\exp}$ as in 1° of the proof of Theorem 8.1). The case $n = 1$ in Theorem 8.1 follows from 4° of the proof of Theorem 8.1. The rest is straightforward or follows from Theorem 9.3. □

We used above the following result.

Lemma 9.7. *Corollary 6.3 holds even if we replace $\mathcal{F}(\mathcal{M}_+^{(n^+)})_{\mathbb{R}}$ by $\mathcal{F}(\mathcal{M}_+^{(n^+),\exp})_{\mathbb{R}}$.*
Corollary 5.10 holds even if we replace $\mathcal{F}\ell^1(\mathbb{N}^n;\mathbb{R})$ by $\mathcal{F}\ell^{1,\exp}(\mathbb{N}^n;\mathbb{R})$.

As obvious from the proofs, instead of $\mathcal{F}(\mathcal{M}_+^{(n^+),\exp})_{\mathbb{R}}$ we could use above any other set containing g', f_j' $(j = 1,\dots,k)$ and a left inverse of (f',g').

Proof. By Remark 9.2, $a : \mathbb{N}^n \to \mathbb{C}$ is in $\ell^{1,\exp}(\mathbb{N}^n)$ iff $\widehat{a}$ converges absolutely on a neighborhood of $\overline{\mathbb{D}}^n$. The functions f_j and g of Lemma 5.9 are of this form, so $f_1,\dots,f_n,g \in \mathcal{F}\ell^{1,\exp}(\mathbb{N}^n)$ $\forall j$ (here $m := n$). This proves the latter claim in Lemma 9.7 (see the proof of Corollary 5.10).

The above implies that $f_1',\dots,f_n',g' \in (\mathcal{M}_+^{(2k^+)})^{\exp}$ (hence $f_1',\dots,f_n',g' \in (\mathcal{M}_+^{(2k^+)})_{\mathbb{R}}^{\exp}$) in Lemma 6.2, by the comment on the isomorphism (4) in Remark 9.2. Therefore, also the first claim in Lemma 9.7 holds (see the proof of Corollary 6.3). □

We set $H^{\infty,\exp}(\mathbb{C}_+) := \{f \in H^\infty(\mathbb{C}_+) \colon f(\cdot - \omega) \in H^\infty(\mathbb{C}_+) \text{ for some } \omega > 0\}$.

Proposition 9.8. *We have $\mathrm{bsr}\,H^{\infty,\exp}(\mathbb{C}_+) = 1$, and $\mathrm{bsr}\,H^{\infty,\exp}(\mathbb{C}_+)_{\mathbb{R}} = 2$.*

Proof. For any function $h \in H^\infty(\mathbb{C}_+)$ and any number $\omega \in \mathbb{R}$ we set $h_\omega := h(\cdot + \omega)$. If $(f,g) \in \mathrm{U}_2(H^{\infty,\exp}(\mathbb{C}_+))$, then $(f_{-\omega}, g_{-\omega}) \in \mathrm{U}_2(H^\infty(\mathbb{C}_+))$ for some $\omega > 0$. But $\mathrm{bsr}\,H^\infty = 1$ [36], and so $f_{-\omega} + hg_{-\omega} \in \mathrm{U}_1$ for some $h \in H^\infty(\mathbb{C}_+)$. Hence $f + h_\omega g \in \mathrm{U}_1(H^{\infty,\exp}(\mathbb{C}_+))$ (note that $h_\omega \in H^{\infty,\exp}(\mathbb{C}_+)$). Consequently, $\mathrm{bsr}\,H^{\infty,\exp}(\mathbb{C}_+) = 1$.

If $g(s) = (s-1)(s-3)/(s+1)^2$ and $f(s) = (s-2)/(s+2)$, then f has different signs at the zeros 1 and 3 of g. Hence $(f,g) \in \mathrm{U}_2(H^{\infty,\exp}(\mathbb{C}_+)_\mathbb{R})$ is not reducible in $H^{\infty,\exp}(\mathbb{C}_+)_\mathbb{R}$ (see the paragraph below Lemma 10.1). Thus $\operatorname{bsr} H^{\infty,\exp}(\mathbb{C}_+)_\mathbb{R} \geq 2$.

That $\operatorname{bsr} H^{\infty,\exp}(\mathbb{C}_+)_\mathbb{R} = 2$ can be seen using the fact that $\operatorname{bsr} H^{\infty,\exp}(\mathbb{C}_+)_\mathbb{R} = 2$ [21]: just modify the first part of the proof: assume that $f \in H^{\infty,\exp}(\mathbb{C}_+)^2_\mathbb{R}$ and that $g \in H^{\infty,\exp}(\mathbb{C}_+)_\mathbb{R}$ etc. □

Recall from Theorem 9.4 that $\operatorname{bsr} H^{\infty,\exp}(\mathbb{D}^n) = \lfloor n/2 \rfloor + 1$ and $\operatorname{bsr} H^{\infty,\exp}(\mathbb{D}^n)_\mathbb{R} = n+1$ (since $H^{\infty,\exp}(\mathbb{D}^n) = \mathrm{A}^{\exp}(\mathbb{D}^n)$).

The reducible elements $(f,g) \in \mathrm{U}_2(\mathcal{A}^{\exp}_\mathbb{R})$ are characterized in Remark 10.6 for many of our real exponential algebras $\mathcal{A}^{\exp}_\mathbb{R}$.

10. Reducible elements of $\ell^1(\mathbb{N};\mathbb{R})$, $\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})$, and other real algebras

A (coprime) pair $(f,g) \in \mathrm{U}_2$ is reducible iff $f + hg \in \mathrm{U}_1$ for some h. All of our real causal algebras have bsr ≥ 2, which means that some coprime pairs are not reducible. Even worse, as noted below 6.2, some pairs generated by two independent positive delays are not reducible even in AP. However, in many real algebras, the coprime pairs that are reducible are exactly those that have the "parity interlacing property" defined later below. That is the subject of this subsection.

The *half-plane algebra* $A(\mathbb{C}_+)$ is the sup-normed Banach algebra of those continuous functions $\overline{\mathbb{C}_+} \cup \{\infty\} \to \mathbb{C}$ that are holomorphic on $\mathbb{C}_+$. It is obviously isometrically isomorphic to $A(\mathbb{D})$ through the Cayley transform $f(\cdot) \mapsto f((1-\cdot)/(1+\cdot))$.

Let $(f,g) \in \mathrm{U}_2(A(\mathbb{C}_+)_\mathbb{R})$. We say that (f,g) has the *parity interlacing property* if f has the same sign at real zeros $\{r \in \mathbb{R}_+ \cup \{\infty\} \colon g(r) = 0\}$ of g. An equivalent condition is that f has an even number of zeros (counting multiplicities) between any real zeros of g. Not all unimodular pairs are reducible (that is, $\operatorname{bsr} A(\mathbb{C}_+)_\mathbb{R} > 1$), but the above property characterizes reducible pairs, as shown in [41] (for $A(\mathbb{D})$) and restated below.

Lemma 10.1. *A pair $(f,g) \in \mathrm{U}_2(A(\mathbb{C}_+)_\mathbb{R})$ is reducible iff f has the same sign at each zero of g on $\mathbb{R}_+ \cup \{+\infty\}$.*

So then $f + hg \in \mathrm{U}_1$ for some $h \in A(\mathbb{C}_+)_\mathbb{R}$. The necessity of the parity interlacing property is obvious, because $f + hg$ is real and nonzero on $\mathbb{R}_+$ and hence must have a constant sign.

One easily verifies that an element of $\mathcal{M}_+ + \mathrm{L}^1_+$ or of $\ell^1(\mathbb{N}^n)$ is real-valued iff its transform is real-symmetric ($\widehat{f}(\overline{z}) = \overline{\widehat{f}(z)}$), so, for example, $\mathcal{F}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R})) = \mathcal{F}(\mathbb{C}\delta_0 + \mathrm{L}^1_+) \cap A(\mathbb{C}_+)_\mathbb{R}$. Therefore, we get the following corollary.

Corollary 10.2. *A pair $(f,g) \in \mathcal{F}\mathrm{U}_2(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}))$ is reducible iff f has the same sign at each zero of g on $\mathbb{R}_+ \cup \{+\infty\}$.*

Proof. The necessity of the property follows as above. Now assume that (f,g) has the property. By Lemma 10.1, there exists $h \in A(\mathbb{C}_+)_\mathbb{R}$ such that $f + gh \in \mathrm{U}_1(A(\mathbb{C}_+)_\mathbb{R})$, that is, $\inf_{\mathbb{C}_+} |f + gh| > 0$ (Theorem 4.5).

Since real-symmetric polynomials are dense in $A(\mathbb{D})_\mathbb{R}$ (Lemma A.1), their Cayley transforms (that is, real-symmetric rational $A(\mathbb{C}_+)_\mathbb{R}$ functions) are dense in $A(\mathbb{C}_+)_\mathbb{R}$. Consequently, $\inf_{\mathbb{C}_+} |f + gh'| > 0$ for some real-symmetric rational $h' \in A(\mathbb{C}_+)_\mathbb{R}$. By Lemma 5.19 (and the fact that $\mathcal{F}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}))$ consists of the real-symmetric elements of $\mathcal{F}(\mathbb{C}\delta_0 + \mathrm{L}^1_+)$), we have $h' \in \mathcal{F}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}))$. But $f + gh' \in \mathrm{U}_1(\mathcal{F}\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}))$, from Theorem 4.5. □

Corollary 10.3. *A pair* $(f,g) \in \mathcal{F}\mathrm{U}_2(\ell^1(\mathbb{N};\mathbb{R}))$ *is reducible iff* f *has the same sign at each zero of* g *on* $[-1,1]$.

Here $\mathcal{F}\ell^1(\mathbb{N})$ can obviously be replaced by any full subring $\mathcal{A} \subset A(\mathbb{D})$ containing all polynomials. The proof is analogous to that of Corollary 10.2 and hence omitted.

The isomorphism of Remark 2.1 extends Corollary 10.3 to $(\mathcal{M}_+^{(1^+)})_{\mathbb{R}}$; we write this out below. If $\mu = \sum_{k=0}^{\infty} a_k \delta_{kT} \in (\mathcal{M}_+^{(1^+)})_{\mathbb{R}}$, then obviously $\widehat{\mu}(r) = \widehat{a}(\mathrm{e}^{-r})$ and $\widehat{\mu}(i/T + r) = \widehat{a}(-\mathrm{e}^{-r})$ for every $r \geq 0$. By Remark 2.1, this and Corollary 10.3 imply the following.

Corollary 10.4. *A pair* $(f,g) \in \mathcal{F}\mathrm{U}_2((\mathcal{M}_+^{(1^+)})_{\mathbb{R}})$ *is reducible iff* f *has the same sign at each zero of* g *on* $\mathbb{R}_+ \cup \{+\infty\} \cup (i/T + \mathbb{R}_+)$.

The part "$\cup(i/T + \mathbb{R}_+)$" cannot be removed from the above corollary: set $T := 1$; for $f(s) = \mathrm{e}^{-s},\ g(s) = \mathrm{e}^{-2s} - 1/4$ the pair $(f,g) \in \mathrm{U}_2$ is not reducible although g has only one zero on $\mathbb{R}_+ \cup \{+\infty\}$, because f has a different sign at the zero on $i + \mathbb{R}_+$.

Corollary 10.5. *A pair* $(f,g) \in \mathcal{F}\mathrm{U}_2((\mathcal{M}_+^{(1^+)} + \mathrm{L}^1_+)_{\mathbb{R}})$ *is reducible iff* f *has the same sign at each zero of* g *on* $\mathbb{R}_+ \cup \{+\infty\}$ *and the discrete part of* (f,g) *is reducible (see Corollary 10.4).*

Proof. As before, the parity interlacing property is necessary. By Lemma 4.1, also the latter condition is necessary. Now assume that both conditions hold. Write $f = f_{\mathrm{d}} + f_{\mathrm{a}}$, $g = g_{\mathrm{d}} + g_{\mathrm{a}}$. Since the discrete part $(f_{\mathrm{d}}, g_{\mathrm{d}})$ is reducible, we have $w := f_{\mathrm{d}} + h_{\mathrm{d}} g_{\mathrm{d}} \in \mathcal{F}\mathrm{U}_1((\mathcal{M}_+^{(1^+)})_{\mathbb{R}})$ for some $h_{\mathrm{d}} \in \mathcal{F}(\mathcal{M}_+^{(1^+)})_{\mathbb{R}}$. Set $F := w^{-1}(f + h_{\mathrm{d}} g) = 1 + w^{-1}(f_{\mathrm{a}} + h_{\mathrm{d}} g_{\mathrm{a}}) \in \mathcal{F}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}))$. From Lemma 3.10, it follows that $(F,g) \in \mathrm{U}_2$, that is, $(F,g) \neq 0$ on $\overline{\mathbb{C}_+} \cup \{\infty\}$ (Theorem 4.5). Hence $(F, \phi g) \in \mathrm{U}_2$, where $\phi(s) := 1/(s+1)$ (because $F(+\infty) = 1$). (By Lemma 5.19, $\phi \in \mathcal{F}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}))$.) Set $F' := F$. Since $w \in \mathrm{U}_1$ has a constant sign $p = \pm 1$ on $\mathbb{R}_+ \cup \{+\infty\}$, the sign of F on the finite real zeros of g (or of ϕg) equals p times that of f, that is, $(F', \phi g)$ has the parity interlacing property. (Unless $g(+\infty) \neq 0$ and $g(R) = 0$ for some $R \in (0,\infty)$ and f and w have different signs at R; in this case set $F' := -F$ instead; then $\operatorname{sgn} F'(+\infty) = -1$, which equals $-p \operatorname{sgn} f(R) = \operatorname{sgn} F'(R)$ and so $(F', \phi g)$ again has this property.) Thus $F' + h_0 \phi g \in \mathrm{U}_1$ for some $h_0 \in \mathcal{F}(\mathbb{R}\delta_0 + \mathrm{L}^1(\mathbb{R}_+;\mathbb{R}))$, by 10.2. From Lemma 3.10, it follows that (f,g) is reducible (namely $f + (h_{\mathrm{d}} + w h_0 \phi) g \in \mathrm{U}_1$ (or $f + (h_{\mathrm{d}} - w h_0 \phi) g \in \mathrm{U}_1$ if $F' = -F$)). □

From the proofs (in particular from that of Corollary 10.2), we observe the following, which can alternatively also be concluded by density (Lemma A.1).

Remark 10.6 ($\mathcal{A}_{\mathbb{R}}^{\exp}$)**.** The element h such that $f + hg \in \mathrm{U}_1$ can be chosen to be rational in Lemma 10.1 and in Corollary 10.2, a polynomial in Corollary 10.3, a finite sum $\sum_{k=1}^{n} h_n \mathrm{e}^{-\cdot nT}$ in Corollary 10.4, and such a finite sum plus a rational function in Corollary 10.5.

Therefore, the algebras in Lemma 10.1 and in Corollaries 10.2–10.5 can be replaced by the corresponding exponential algebras (defined in §9).

Proof. The first paragraph obviously holds.

Let then $(f,g) \in \mathrm{U}_2(\mathcal{A}^{\exp})$, where $\mathcal{A}^{\exp}$ is the exponential form of the algebra in some of the results. Since $\mathrm{U}_1(\mathcal{A}^{\exp}) \subset \mathrm{U}_1(\mathcal{A})$, the corresponding parity interlacing property is still necessary; by the first paragraph it is also sufficient for the existence of $h \in \mathcal{A}^{\exp}$ such that $f + hg \in \mathrm{U}_1(\mathcal{A}^{\exp})$. But $f + hg \in \mathrm{U}_1(\mathcal{A}^{\exp})$ iff $f + hg \in \mathrm{U}_1(\mathcal{A})$, because $\mathcal{A}^{\exp}$ is full in $\mathcal{A}$, by Theorem 9.3, so we are done. □

A result for $H^\infty_{\mathbb{R}}$, analogous to that in Lemma 10.1, is given in [40].

An example of extending this characterization to the matrix-valued case (with f rational and $\mathcal{F}^{-1}g$ a measure) is given in [32], or in the rational case also in [38, p. 118]. In these results $\det f$ should have the same sign at all real zeros of g. We also mention that $\operatorname{bsr} \mathcal{A}^{n\times n} = \lfloor -(\operatorname{bsr} \mathcal{A} - 1)/n \rfloor + 1$ for an arbitrary ring $\mathcal{A}$ [37, Theorem 3], and for a Banach algebra $\mathcal{A}$ with a continuous involution, $\operatorname{tsr} \mathcal{A}^{n\times n} = \lfloor (\operatorname{tsr} \mathcal{A} - 1)/n \rfloor + 2$; see [23, Theorem 6.1].

10.1. **Control-theoretic consequences.** The connection between Bass and topological stable ranks and different forms of stabilization of linear systems is best explained in [22], whose results have raised the question of the stable ranks of many of the algebras treated in this article. A wealthy background on the theme is provided by [38]. Therefore, here we mainly make some supplementing observations.

Many of our algebras appear frequently in control theoretic literature, such as [38], [39] (which treats $\ell^1(\mathbb{N})$, $\ell^1(\mathbb{N}^2)$, $\ell^{1,\exp}(\mathbb{N})$, $\mathcal{M}_+ + \mathrm{L}^1_+$, and $(\mathcal{M}_+ + \mathrm{L}^1(\mathbb{R}_+))^{\exp}$), [4] and [22]; see, e.g., [9] for many practical examples. In many cases, the transfer functions actually lie in (the fields of fractions of) smaller algebras (than those studied in the references), which have lower stable ranks and hence lead to more powerful results. This and the many $\operatorname{bsr} = \infty$ results in this article emphasize the importance the other algebras (e.g., in Table 1).

One of the motivations for reducing a coprime pair $(f, g) \in \mathrm{U}_2(\mathcal{A})$, that is, for finding $h \in \mathcal{A}$ such that $f + hg$ is invertible, comes from control theory. Indeed, if $\mathcal{A}$ models *stable* transfer functions and the *transfer function* of a system is given by g/f, then such a function h exists iff the system can be *strongly stabilized* (stabilized by a stable controller, namely $-h \in \mathcal{A}$). [38] [22].

The advantages of strong stabilization are also explained in [38]. They include weaker sensitivity to disturbances, better ability to track reference inputs, the possibility to use the two-stage stabilization procedure, as well as applications in simultaneous stabilization, which leads to better robustness against structural changes including nonlinearities and loss of components. [38]

In continuous-time (respectively, discrete-time) applications, often $\mathcal{A}$ equals $\mathcal{F}(\mathcal{M}_+ + \mathrm{L}^1_+)$ (respectively, $\mathcal{F}\ell^1(\mathbb{N})$) or some subset of it (possibly real, which motivates this section). If $\operatorname{bsr} \mathcal{A} = 1$, then any two plants of same dimensions, admitting doubly coprime factorizations (cf. Subsection 10.2 below), can be stabilized by a controller [22, Corollary 6.7]. As the proof (whose origin is in [38]) of that result shows, this simultaneous stabilization can be reduced to a strong stabilization problem in the same algebra, so also the results of this section apply to simultaneous stabilization.

Some control-theoretic applications of other bsr and tsr results (for example, of $\operatorname{bsr} \leq 2$ or of $\operatorname{tsr} \leq 2$) are explained in [22], where it is also shown how scalar-valued results readily extend to the matrix case. If, for example, $\operatorname{bsr} \mathcal{A} = 1$, then $\operatorname{bsr} \mathcal{A}^{n\times n} = 1$, by [37, Theorem 3]. It easily follows (first extend the matrices to squares and later discard unnecessary blocks) that if $f \in \mathcal{A}^{n\times n}$ and $g \in \mathcal{A}^{m\times n}$ are coprime, then gf^{-1} is strongly stabilizable (that is, there exists $h \in \mathcal{A}^{n\times m}$ such that $f + hg \in \mathrm{U}_1(\mathcal{A}^{n\times n})$. A generalization of this result is given in [22]. Moreover, the above method is constructive if the corresponding scalar result is constructive (as they mostly are).

We also note that most of the applications of stable rank and reduction results are robust with respect to errors, because U_n is open (Lemma 3.4). For example, strong stabilization is robust with respect to small errors in f, g and h (we still have $f + hg \in \mathrm{U}_1(\mathcal{A})$). Moreover, since most of our algebras are full in the corresponding H^∞ algebra (Theorem 4.3), the applications are actually robust in H^∞, that is, the true functions (data or solutions) need not even lie in $\mathcal{A}$ at all, it suffices that they are close to our models in the supremum norm if we are satisfied by the corresponding properties in H^∞ (for example, for strong stabilization in H^∞). Furthermore, in general the construction of $h \in H^\infty$ such that $f + hg \in \mathrm{U}_1(H^\infty)$ is terribly difficult [36], so there is a huge advantage of being able to make any constructions in a nice algebra where more elementary constructive results are available. Naturally, the corresponding simple model for h is also desirable.

10.2. **Existence of coprime factorizations.** Above we considered coprime factorizations g/f. Often transfer functions are given as fractions g/f with $g, f \in \mathcal{A}$, $f \neq 0$ that are not necessarily coprime ($\in \mathrm{U}_2(\mathcal{A})$). To be able to apply the numerous results for coprime fractions in the literature, such as those in this article, one then wants to find an equivalent coprime factorization. We treat this problem in the remark below.

Remark 10.7. Let $f, g \in \mathcal{A}$, $f \neq 0$, where $\mathcal{A}$ is a commutative ring with no zero divisors (that is, $a, b \in \mathcal{A}$, $ab = 0$, $b \neq 0 \Rightarrow a = 0$). Note that all (non-matrix) algebras defined in this article are of that form. We call "$g/f = w/v$" a *coprime factorization* of "g/f" if $(v, w) \in \mathrm{U}_2(\mathcal{A})$ and $gv = fw$.

The ring $\mathcal{A}$ is a so-called *Bézout domain* iff all such fractions have a coprime factorization [38, p. 332].

Using the corona theorems of this article and methods similar to those in [39] and [18], one can show that $\mathcal{M}_+^{1,\exp}$, $(\mathcal{M}_+^{(1^+)})_{\mathbb{R}}^{\exp}$, $\mathrm{A}^{\exp}(\mathbb{D}) = H^{\infty,\exp}(\mathbb{D})$, $\mathrm{A}^{\exp}(\mathbb{D})_{\mathbb{R}} = H^{\infty,\exp}(\mathbb{D})_{\mathbb{R}}$, $\mathrm{C}^{\exp}(\mathbb{T})$, $\mathrm{C}^{\exp}(\mathbb{T})_{\mathbb{R}}$, $\ell^{1,\exp}(\mathbb{N})$ and $\ell^{1,\exp}(\mathbb{N};\mathbb{R})$ are Bézout domains but the none of the other algebras defined in this article are Bézout domains.

However, under the additional assumption that the denominator f is *bounded below at infinity*, that is, $|f(z)| \geq \epsilon$ ($z \in \mathbb{C}_+$, $|z| > R$) for some $\epsilon, R > 0$, then f/g does have a coprime factorization provided that $\mathcal{A}$ equals $H^{\infty,\exp}(\mathbb{C}_+)$, $H_{\mathbb{R}}^{\infty,\exp}(\mathbb{C}_+)$, $(\mathcal{A}' + \mathrm{L}^1(\mathbb{R}_+))^{\exp}$, or $(\mathcal{A}' + \mathrm{L}^1(\mathbb{R}_+))_{\mathbb{R}}^{\exp}$, where $\mathcal{A}'$ is a subalgebra of $\mathcal{M}_+$ and the corona theorem holds for $\mathcal{A}'$.

We omit the proofs. In all positive results above, a suitable coprime factorization can be obtained by dividing out the zeros of (f, g) by a suitable rational function (then use the corona Theorem 9.3). The proofs of the negative results are mostly similar to those given in the two references.

An analogous claim holds also in the matrix-valued case (cf. p. 885 and Theorem 2.1 of [39]).

The fact that $\ell^{1,\exp}(\mathbb{N})$ is a Bézout domain (and $\ell^1(\mathbb{N})$ and $\ell^1(\mathbb{N}^2)$ are not) was proved in [39]. For $(\mathcal{M}_+ + \mathrm{L}^1(\mathbb{R}_+))^{\exp}$ the existence of a coprime factorization was shown in [4] assuming that f is bounded below at infinity. Some of the negative results were established in [18].

Appendix A. Auxiliary results

In this appendix we present some technical results commonly used in our proofs.

First we describe important dense subsets of our algebras.

Lemma A.1 (density)**.** *In each ℓ^1 algebra, elements with finite support are dense. Consequently, finite linear combinations of (included) δ_r's are dense in each $\mathcal{M}_*^*$ algebra (see Remark 2.1). Their Laplace transforms are dense in* AP, AP$_+$ *and* AP$_S$. *Continuous functions with compact supports are dense in each* L^1 *space in this article. Polynomials are dense* *in $A(\mathbb{D}^n)$, by Taylor's Theorem. Polynomials in z_k's and in z_k^{-1}'s are dense in $C(\mathbb{T}^n)$, by the Stone–Weierstrass theorem. Corresponding claims also hold for corresponding real algebras.*

So for example, real polynomials are dense in $A(\mathbb{D}^n)_\mathbb{R}$ and real elements of $\ell^1(\mathbb{R})$ with finite support are dense in $\ell^1(\mathbb{R};\mathbb{R})$. Obviously, Lemma A.1 holds for the corresponding exponential algebras too. By Lemma 5.19, rational functions are dense in $\mathcal{F}(\mathbb{C}\delta_0 + \mathrm{L}^1)$.

Proof. This is straightforward except for the real case. But that follows from corresponding complex claims (or as above), for example, if $h \in C(\mathbb{T}^n)_\mathbb{R}$ and some polynomials p_n converge uniformly to h on $\mathbb{D}$, then $\|(p_n)_R - h\|_\infty = \|(p_n - h)_R\|_\infty \to 0$, by Lemma 4.4. □

We recall from [26, p. 153] that a differentiable image of a null set is a null set.

Lemma A.2. *If $V \subset \mathbb{R}^n$ is open, $f : V \to \mathbb{R}^n$ is differentiable, $\Omega \subset V$, and $m(\Omega) = 0$, then $m(f[\Omega]) = 0$; hence then $\mathbb{R}^n \setminus f[\Omega]$ is dense in $\mathbb{R}^n$.*

If $\Omega \subset \mathbb{R}^k$ is open, $k < n$, and $f : \Omega \to \mathbb{R}^n$ is differentiable, then $m(f[\Omega]) = 0$; hence then $\mathbb{R}^n \setminus f[\Omega]$ is dense in $\mathbb{R}^n$.

To prove the latter claim, apply the former to $g(x_1, \ldots, x_n) := f(x_1, \ldots, x_k)$, $V := \Omega \times \mathbb{R}^{n-k}$, and $\Omega' := \Omega \times \{0\}^{n-k}$.

The following is an obvious consequence of Brouwer's fixed point theorem.

Lemma A.3 (Brouwer)**.** *If $f \in C(\mathbb{R}^n;\mathbb{R}^n)$ is bounded, then $f(x) = x$ for some $x \in \mathbb{R}^n$.*

We recall the Riemann–Lebesgue Lemma.

Lemma A.4 (Riemann–Lebesgue)**.** *If $f \in \mathrm{L}^1(\mathbb{R}_+)$ (respectively, $f \in \mathrm{L}^1(\mathbb{R})$) and $\epsilon > 0$, then there exists $R < \infty$ such that $|f(z)| < \epsilon$ when $z \in \mathbb{C}_+$ (respectively, $z \in i\mathbb{R}$) and $|z| \geq R$.*

The next result says that if $f \in C^1$ has a zero $f(a) = 0$ with invertible derivative $f'(a)$, then for small $g \in C$, $f + g$ also has a zero (near a).

Lemma A.5. *Let $V \subset \mathbb{R}^n$ be open and $a \in V$. If $f \in C^1(V;\mathbb{R}^n)$, $f(a) = b$ and $f'(a)$ is invertible, then there exists $\delta > 0$ such that for every $g \in C(V;\mathbb{R}^n)$ satisfying $\|g\|_\infty < \delta$, there holds that $b \in (f + g)[V]$.*

Assume, in addition, that $M'' := \sup_V \|f''\| < \infty$. If $M := \|f'(a)^{-1}\|$, then we can have $\delta = 1/4M^2M''$ above.

Proof.

1° We can and will assume that $a = 0 = b$. By the Inverse Function Theorem [25], there exists $r > 0$ and an open $W \subset \mathbb{R}^n$ such that $B_r \subset V$, f is one-to-one on B_r with inverse $h \in C^1(W;B_r)$, where $W := f[B_r]$, $B_r := \{x \in \mathbb{R}^n \colon |x| < r\}$. Fix some $\epsilon \in (0, r)$. Since $h(0) = 0$ (that is, $h(b) = a$), there exists $\delta > 0$ such that $|h(x)| \leq \epsilon$ when $|x| \leq \delta$.

Assume that $\|g\|_\infty \leq \delta$. The function $G := h \circ -g \in C(V;\mathbb{R}^n)$ satisfies $|G(x)| \leq \epsilon$ for $x \in V$, so G maps $\bar{B}_\epsilon \to \bar{B}_\epsilon$. From Brouwer's fixed point theorem, it follows that $G(x) = x$ for some $x \in \bar{B}_\epsilon$. But $f(x) = f(G(x)) = f(h(-g(x))) = -g(x)$, that is, $f(x) + g(x) = 0$, as desired.

2° From the proof of [25, Theorem 9.24], we observe that $r := 1/2MM''$ will do above. But $|f(x)| \geq 0 + |f'(a)x| - M''|x|^2/2 \geq |x|/\|A^{-1}\| - M''|x|^2/2 > |x|M^{-1}(1 - MM''r/2) \geq \epsilon M^{-1}3/4 = \delta$ when $r > |x| \geq \epsilon$. Thus $f[B_r \setminus B_\epsilon] \subset B_\delta^c$. So $h[B_\delta] \subset B_\epsilon \cup B_r$, hence $h[B_\delta] \subset B_\epsilon$ (being connected and containing $h(0) = 0$), so this δ will do. Since $\epsilon < r$ was arbitrary, any $\delta < rM^{-1}3/4 = 3/8M^2M''$ will do.

□

References

[1] R. Arens and I.M. Singer. Generalized analytic functions. *Transactions of the American Mathematical Society*, 81:379-393, 1956.

[2] C. Badea. The stable rank of topological algebras and a problem of R. G. Swan. *Journal of Functional Analysis*, no. 1, 160:42-78, 1998.

[3] H. Bass. K-theory and stable algebra. *Publications Mathématiques de L'I.H.É.S.*, 22:5-60, 1964.

[4] F.M. Callier and C.A. Desoer. An algebra of transfer functions for distributed liner time-invariant systems. *IEEE Transactions on Circuits and Systems*, no. 9, 25:651-662, 1978.

[5] K.F. Clancey and I. Gohberg. *Factorization of matrix functions and singular integral operators*. Operator Theory: Advances and Applications, vol. 3, Birkhäuser-Verlag, Basel Boston Berlin, 1981.

[6] G. Corach and A.R. Larotonda. Stable range in Banach algebras. *Journal of Pure and Applied Algebra*, 32:289-300, 1984.

[7] G. Corach and F.D. Suárez. Extension problems and stable rank in commutative Banach algebras. *Topology and its Applications*, 21:1-8, 1985.

[8] G. Corach and F.D. Suárez. Dense morphisms in commutative Banach algebras. *Transactions of the American Mathematical Society*, no. 2, 304:537-547, 1987.

[9] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Texts in Applied Mathematics 21, Springer, 1995.

[10] T.W. Dawson and J.F. Feinstein. On the denseness of the invertible group in Banach algebras. *Proceedings of the American Mathematical Society*, no. 9, 131:2831-2839, 2003.

[11] P.L. Duren. *The Theory of H^p Spaces.* Academic Press, 1970.

[12] T.W. Gamelin. *Uniform Algebras.* Prentice-Hall, 1969.

[13] I. Gelfand, D. Raikov and G. Shilov. *Commutative Normed Rings.* Chelsea Publishing Company, 1964.

[14] R.H. Herman and L.N. Vaserstein. The stable range of C^*-algebras. *Inventiones Mathematicae*, 77:553-555, 1984.

[15] Einar Hille and Ralph S. Phillips. *Functional Analysis and Semi-Groups.* AMS, Providence, revised edition, 1957.

[16] M.W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press, 1974.

[17] P.W. Jones, D. Marshall and T. Wolff. Stable rank of the disc algebra. *Proceedings of the American Mathematical Society*, 96:603-604, 1986.

[18] H. Logemann. Finitely generated ideals in certain algebras of transfer functions for infinite-dimensional systems. *Int. J. Control*, 45:247–250, 1987.

[19] K.M. Mikkola. *Infinite Dimensional Linear Systems, Optimal Control and Riccati Equations.* Ph.D. thesis, Helsinki University of Technology, Finland, 2002. Available at `http://www.math.hut.fi/reports`.

[20] R. Mortini. An example of a subalgebra of H^∞ on the unit disk whose stable rank is not finite. *Studia Mathematica*, no. 3, 103:275-281, 1992.

[21] R. Mortini and B. Wick. The Bass and topological stable Ranks for real H^∞ and the real disc algebra. To appear in *J. Reine Angew. Math.*. Available at `www.math.sc.edu/~bwick/papers/realstablerank.pdf` .

[22] A. Quadrat. On a general structure of the stabilizing controllers based on stable range. *SIAM Journal on Control and Optimization*, no. 6, 42:2264-2285, 2004.

[23] M.A. Rieffel. Dimension and stable rank in the K-theory of C^*-algebras. *Proceedings of the London Mathematical Society*, no. 2, 46:301-333, 1983.

[24] L. Rodman and I.M. Spitkovsky. Almost periodic factorization and corona theorem. *Indiana University Mathematics Journal*, no. 4, 47:1243-1256, 1998.

[25] W. Rudin. *Principles of Mathematical Analysis*, Third Edition. McGraw-Hill, 1976.

[26] W. Rudin. *Real and Complex Analysis*, Third Edition. McGraw-Hill, 1987.

[27] W. Rudin. *Functional analysis. Second edition.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

[28] R. Rupp. Stable rank of subalgebras of the ball algebras. *Proceedings of the American Mathematical Society*, no. 3, 109:781-786, 1990.

[29] R. Rupp. Stable rank of holomorphic function algebras. *Studia Mathematica*, no. 2, 97:85-90, 1990.

[30] R. Rupp. Stable rank of finitely generated algebras. *Archiv der Mathematik*, no. 5, 55:438-444, 1990.
[31] R. Rupp and A.J. Sasane. On the stable rank of the real disk algebra. *Preprint*, 2006. Available at: `http://www.maths.lse.ac.uk/Personal/amol/RS_AMS.pdf` .
[32] O.J. Staffans. Stabilization of a distributed system with a stable compensator. *Math. Control Signals Systems*, no. 1, 5:1–22, 1992.
[33] F.D. Suárez. The algebra of almost periodic functions has infinite topological stable rank. *Proceedings of the American Mathematical Society*, no. 3, 124:873-876, 1996.
[34] F.D. Suárez. Trivial Gleason parts and the topological stable rank of H^∞. *American Journal of Mathematics*, no. 4, 118:879-904, 1996.
[35] R.G. Swan. Topological examples of projective modules. *Transactions of the American Mathematical Society*, 230:201-234, 1977.
[36] S.R. Treil. The stable rank of the algebra H^∞ equals 1. *Journal of Functional Analysis*, 109:130-154, 1992.
[37] L.N. Vaseršteĭn. The stable range of rings and the dimension of topological spaces. (Russian) *Akademija Nauk SSSR. Funkcional. Analiz i ego Priloženija.*, no. 2, 5:17-27, 1971. English translation in [*Functional Anal. Appl.*, 5:102-110, 1971].
[38] M. Vidyasagar. *Control System Synthesis: a Factorization Approach.* MIT Press, 1985.
[39] M. Vidyasagar, H. Schneider and B. Francis. Algebraic and topological aspects of feedback stabilization. *IEEE Trans. Automat. Control*, no. 4, 27:880–894, 1982.
[40] B.D. Wick. Stabilization in $H^\infty_{\mathbb{R}}(\mathbb{D})$. *Preprint*, 2006. Available at `www.math.sc.edu/~bwick/papers/realhinfty.pdf` .
[41] B.D. Wick. A note about stabilization in $A_{\mathbb{R}}(\mathbb{D})$. To appear in *Math. Nachr.*, 2007. Available at `www.math.sc.edu/~bwick/papers/realdiscalgebra.pdf` .

Institute of Mathematics, Helsinki University of Technology, Box 1100, 02015 HUT, Finland.
E-mail address: `Kalle.Mikkola@iki.fi`

Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom.
E-mail address: `A.J.Sasane@lse.ac.uk`