QUASIREGULAR MAPS AND THE CONDUCTIVITY EQUATION IN THE HEISENBERG GROUP

ANTON ISOPOUSSU, KIRSI PELTONEN, AND JEREMY T. TYSON

Abstract. We show that the interplay between the planar Beltrami equation governing quasiconformal and quasiregular mappings and Calderón's conductivity equation in impedance tomography admits a counterpart in the setting of the first Heisenberg group equipped with its canonical sub-Riemannian structure.

1. Introduction

In this paper we extend to the first Heisenberg group endowed with its standard sub-Riemannian (Carnot-Carathéodory) structure, some aspects of geometric function theory and elliptic PDE in the plane. Especially we show that the beautiful bridge between planar quasiconformal mappings governed by the Beltrami equation and the problem of impedance tomography as formulated by Calderón materializes analogously as in the plane [2]. Although the situation is much more rigid in the Heisenberg group than it is in the plane, it is interesting that—at least formally—the successful planar methods outlined in [1] have natural sub-Riemannian counterparts.

A large part of the motivation for this paper comes from applications in engineering and medical diagnostics. We presume that the first Heisenberg group is a potential local model for studies related to electromagnetism and anisotropic media. It is also interesting to observe that biharmonic equations rise in this setting.

We introduce certain nonlinear PDE systems, the so-called *conductivity equations*, which can be written in vector form as

$$\nabla_{\mathbb{H}} \times \big(J\sigma\nabla_{\mathbb{H}}u\big) = 0, \tag{1.1}$$

for a real valued function u on a domain in the Heisenberg group $\mathbb{H}$ and a horizontal conductivity matrix σ. Here $\nabla_{\mathbb{H}}u$ denotes the horizontal gradient of u, while J denotes the standard planar skew-involution acting on the horizontal tangent bundle $H\mathbb{H}$. The equation (1.1) is third-order with respect to horizontal derivatives. In Theorem 3.3, we show that the components of a sufficiently regular quasiregular map of $\mathbb{H}$ satisfy such a system. These systems are obtained from the usual Beltrami equation via the identification of the complex dilatation with a measurable conformal structure. In the Heisenberg setting the conductivity equation is formulated using the notion of the horizontal curl $\nabla_{\mathbb{H}} \times V$ of a horizontal vector field V, which has been recently studied by Franchi et al. [8], [9].

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We also present (see Theorem 3.5) an equivalent formulation in terms of differential forms:

$$\delta_c(\sigma \mathrm{d}_c u \wedge \tau) = 0, \tag{1.2}$$

where d_c is the horizontal exterior derivative acting on functions, δ_c is the formal horizontal L^2-adjoint acting here on 2-forms, and τ is the standard contact form defining the horizontal distribution in $\mathbb{H}$. Equation (1.2) allows us to write down a weak formulation of the conductivity equation (1.1). Using this language, we show how to introduce div-curl couples analogously as for the Hodge $*$ method in the plane case as explained in [1, Chapter 16].

More to the point, we show that the preceding process can be reversed. To each solution f to a given conductivity equation (1.1), we associate a conjugate solution g to the same equation. The pair (f, g) then satisfies a conductivity system. Provided a certain compatibility condition is satisfied (see (4.5)), we further associate to the pair (f, g) a third function h so that the triple (f, g, h) defines a contact map of $\mathbb{H}$. If the original conductivity equation satisfies an appropriate ellipticity bound, then the induced map $F = (f, g, h)$ is quasiregular. This provides a new method for constructing quasiregular maps of the Heisenberg group.

Although the standard complex Beltrami equation can be written in real form

$$D_{\mathbb{H}}F(p)^T D_{\mathbb{H}}F(p) = \lambda(p)\sigma(p) \tag{1.3}$$

also in the higher dimensional Heisenberg groups $\mathbb{H}_n$, $n > 1$, we do not know if there is any relation between the horizontal $2n \times 2n$ conductivity matrix and the complex antilinear mapping μ acting on the holomorphic vectors of the complexified horizontal bundle, as described in [12]. Mappings F as in (1.3) act on domains of $\mathbb{H}_n$ and the conformal factor λ coincides with $(\det D_{\mathbb{H}}F(p))^{1/n}$ if σ is considered as a conformal structure and the normalization $\det \sigma = 1$ is assumed. It is also not yet understood in higher dimensional Euclidean spaces if there is any connection between solutions to conductivity equations and quasiconformal mappings satisfying the real Beltrami equation. In $\mathbb{H}_n$ the relevant equations could be those that are formulated in $\mathbb{R}^{2n}$ corresponding to horizontal operations.

One could also consider more general Beltrami equations containing complex dilatations μ and ν as in [1, Theorem 16.1.6] and study their generalizations to the first Heisenberg group. We return to this and other aspects of this study elsewhere.

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2. Background material

2.1. **Quasiconformal and quasiregular maps in the Heisenberg group.** We denote by $\mathbb{H}$ the first Heisenberg group, with coordinates $p = (x_1, x_2, x_3)$. We also use complex notation $p = (z, x_3)$, where $z = x_1 + \mathbf{i}x_2$. For the basic theory of the sub-Riemannian structure of $\mathbb{H}$ we refer to [5].

A real-valued function u defined on a domain $U \subset \mathbb{H}$ is said to lie in the *horizontal Sobolev space* $W^{1,p}_{\mathbb{H}}(U)$ if all iterated partial derivatives of u with respect to the operators X_1 and X_2 exist weakly as elements of $L^p(U)$. Here we have denoted by $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$

and $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$ the usual basis of horizontal vector fields in $\mathbb{H}$. Replacing L^p with L^p_{loc} yields the *local horizontal Sobolev space* $W^{1,p}_{\mathbb{H},loc}(U)$.

Quasiregular mappings of $\mathbb{H}$ were first considered by Heinonen and Holopainen [10]; the basic regularity assumptions required for the theory were clarified by Dairbekov [6].

Let $U, U' \subset \mathbb{H}$ be domains. We consider maps $F : U \to U'$ which we write in real coordinates as $F = (f_1, f_2, f_3)$. We say that $F \in W^{1,p}_{\mathbb{H},loc}(U)$ if $f_j \in W^{1,p}_{\mathbb{H},loc}(U)$ for $j = 1, 2, 3$.

Definition 2.1 (Heinonen–Holopainen [10]; Dairbekov [6])**.** Let $K \geq 1$. A continuous map $F : U \to U'$ in the local horizontal Sobolev space $W^{1,4}_{\mathbb{H},loc}(U)$ is called K*-quasiregular* if F is a generalized contact map and the dilatation estimate

$$||D_{\mathbb{H}}F(p)||^2 \leq K \det D_{\mathbb{H}}F(p). \tag{2.1}$$

holds for a.e. in $p \in U$.

We recall that F is said to be a *generalized contact map* if

$$\tau_p(X_1F) = \tau_p(X_2F) = 0 \qquad \text{for a.e. } p \in U,$$

where

$$\tau = \mathrm{d}x_3 + 2x_1\,\mathrm{d}x_2 - 2x_2\,\mathrm{d}x_1$$

denotes the standard contact form defining the horizontal distribution in $\mathbb{H}$. The expression $\tau_p(X_jF)$ is shorthand for the action of τ on the vector (X_jf_1, X_jf_2, X_jf_3) at p, i.e.

$$\tau_p(X_jF) = (X_jf_3 + 2f_1\,X_jf_2 - 2f_2\,X_jf_1)(p).$$

The *horizontal differential* $D_{\mathbb{H}}F$ of F at p is the 2×2 matrix

$$D_{\mathbb{H}}F(p) = \begin{pmatrix} X_1f_1(p) & X_2f_1(p) \\ X_1f_2(p) & X_2f_2(p) \end{pmatrix}.$$

The notation $||A||$ denotes the operator norm of a matrix A.

Observe that (2.1) is equivalent to

$$||D_{\mathbb{H}}F(p)||^4 \leq K^2(\det D_{\mathbb{H}}F(p))^2$$

which is a more traditional formulation for the quasiregularity condition on $\mathbb{H}$ (see, e.g., [10] or [6]). A generalized contact map F as above acts on the contact form τ according to the formula

$$F^*\tau = \lambda\tau$$

with $\lambda = \det D_{\mathbb{H}}F$. In this setting, the quantity

$$\lambda \cdot \det D_{\mathbb{H}}F = (\det D_{\mathbb{H}}F)^2$$

represents the volume derivative of the map F. (For C^1 maps, this coincides with the full Jacobian $\det DF$.)

For later use we recall the following result of Dairbekov, see Remark 3 in [6].

Theorem 2.2 (Dairbekov)**.** *Let $F \in W^{1,4}_{\mathbb{H},loc}(U, \mathbb{H})$ be a generalized contact map verifying the dilatation estimate* (2.1) *for a.e. p. Then F is continuous, i.e., F is K-quasiregular.*

We next recall from [11], [13] the formalism of Beltrami differentials on the Heisenberg group. To this end it is convenient to introduce additional notation which is motivated by the appearance of the Heisenberg group as the group of translations of the Siegel upper half space. We will write $F = (f_I, f_3)$, where $f_I = f_1 + \mathbf{i}f_2$. We also use the notation

$f_{II} = f_3 + \mathbf{i} f_4 = f_3 + \mathbf{i}|f_I|^2$. Let us remark that f_4 is also in $W^{1,4}_{\mathbb{H},loc}(U)$ since f_I is continuous and lies in $W^{1,4}_{\mathbb{H},loc}(U)$.

Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc in the complex plane. Then to each quasiregular map F as defined above, there exists a map $\mu : \mathbb{H} \to \mathbb{D}$ with $||\mu||_\infty < 1$ and

$$\overline{Z} f_\ell = \mu\, Z f_\ell, \qquad \ell = I, II. \tag{2.2}$$

Here $Z = \partial_z + \mathbf{i}\bar{z}\partial_{x_3} = \frac{1}{2}(X_1 - \mathbf{i}X_2)$ and $\overline{Z} = \partial_{\bar{z}} - \mathbf{i}z\partial_{x_3} = \frac{1}{2}(X_1 + \mathbf{i}X_2)$. We call μ the *complex dilatation* of F. If F is K-quasiregular, then

$$||\mu||_\infty \le k < 1 \tag{2.3}$$

where

$$k = \frac{K-1}{K+1}. \tag{2.4}$$

Conversely, every continuous map $F : U \to \mathbb{H}$ in $W^{1,4}_{\mathbb{H},loc}$ satisfying (2.2) with μ satisfying (2.3) is K-QR with K and k related by (2.4).

2.2. **Horizontal div, grad, curl and corresponding intrinsic forms.** We introduce the Folland–Stein regularity class $C^k_{\mathbb{H}}(U)$ consisting of real-valued functions u defined on U for which all of the k-fold iterated horizontal partial derivatives $X_{i_1}X_{i_2}\cdots X_{i_k}u$, where $i_1, \ldots, i_k \in \{1,2\}$, exist and are continuous.

We say that a (continuous) vector field V defined on U is *horizontal* if $V(p)$ lies in the horizontal tangent space $H_p\mathbb{H} := \operatorname{span}\{X_1, X_2\}(p)$ for every $p \in U$. The *horizontal gradient* of a function $u \in C^1_{\mathbb{H}}(U)$ is the horizontal vector field

$$\nabla_{\mathbb{H}} u = (X_1 u)X_1 + (X_2 u)X_2. \tag{2.5}$$

The *horizontal divergence* of a $C^1_{\mathbb{H}}$ horizontal vector field $V = a_1X_1 + a_2X_2$ is the function

$$\nabla_{\mathbb{H}} \cdot V = X_1 a_1 + X_2 a_2. \tag{2.6}$$

We now recall the notion of horizontal curl of a horizontal vector field, introduced by Franchi, Tchou and Tesi in [8] and further studied in [9].

Definition 2.3 (Franchi–Tchou–Tesi)**.** Let $V = a_1X_1 + a_2X_2$ be a $C^2_{\mathbb{H}}$ horizontal vector field on U. The *horizontal curl* of V is the horizontal vector field

$$\nabla_{\mathbb{H}} \times V = P_1(V)X_1 + P_2(V)X_2, \tag{2.7}$$

where

$$P_1(V) = P_1(a_1, a_2) = \tfrac{1}{4}\left(X_2X_2a_1 - 2X_2X_1a_2 + X_1X_2a_2\right) \tag{2.8}$$

and

$$P_2(V) = P_2(a_1, a_2) = \tfrac{1}{4}\left(X_1X_1a_2 - 2X_1X_2a_1 + X_2X_1a_1\right). \tag{2.9}$$

Note that the horizontal curl is a **second-order** differential operator in the horizontal partial derivatives X_1 and X_2.

Theorem 2.4 (Franchi–Tchou–Tesi)**.** *(a) For any $C^3_{\mathbb{H}}$ horizontal vector field V,*

$$\nabla_{\mathbb{H}} \cdot (\nabla_{\mathbb{H}} \times V) = 0.$$

(b) For any $u \in C^3_{\mathbb{H}}(U)$, $\nabla_{\mathbb{H}} \times \nabla_{\mathbb{H}} u = 0$. Conversely, if $\nabla_{\mathbb{H}} \times V = 0$ for some $C^2_{\mathbb{H}}$ horizontal vector field V on a simply connected domain U, then there exists $u \in C^3_{\mathbb{H}}(U)$ so that $V = \nabla_{\mathbb{H}} u$.

In this paper the following representation for the horizontal curl operator will play an important role:

$$\nabla_{\mathbb{H}} \times V = JTV - \tfrac{1}{4} J\nabla_{\mathbb{H}}(\nabla_{\mathbb{H}} \cdot JV). \tag{2.10}$$

Here

$$T = \partial_{x_3} = -\tfrac{1}{4}[X_1, X_2] = \tfrac{1}{4}\nabla_{\mathbb{H}} \cdot J\nabla_{\mathbb{H}} \tag{2.11}$$

denotes the vertical (Reeb) vector field in $\mathbb{H}$, while

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

denotes the usual skew-involution of the plane. The operator J naturally acts on horizontal vector fields in the following way: if $V = a_1X_1 + a_2X_2$ is a horizontal vector field, then $JV = -a_2X_1 + a_1X_2$. The notation TV means that the operator T is applied to each of the components of V: if $V = a_1X_1 + a_2X_2$ then $TV = (Ta_1)X_1 + (Ta_2)X_2$.

Remark 2.5. The factor of $\frac{1}{4}$ in (2.8) and (2.9) is merely a normalization. With this factor included one has a version of Stokes' formula on the Heisenberg group which exactly parallels the Euclidean case; see Theorem 5.3 in [8]. This multiplicative factor is irrelevant for the purposes of this paper.

Remark 2.6. From (2.10) and (2.11) we observe the following simple expression for the horizontal curl:

$$\nabla_{\mathbb{H}}\times = \tfrac{1}{4} J[\nabla_{\mathbb{H}} \cdot J, \nabla_{\mathbb{H}}].$$

We will not use this expression in this paper, but record it as a useful mnemonic.

There is another approach to the horizontal curl which uses the language of differential forms. For the purposes of this paper, we briefly sketch the setup for the horizontal differential complex (*Rumin complex*) (E_0^*, d_c) in the first Heisenberg group. Here we follow the explicit representation in [9] based on M. Rumin's theory of intrinsic forms [15], [16].

Denote by (Ω^*, d) the usual de Rham complex of differential forms on $\mathbb{H}$ (identified with $\mathbb{R}^3$). The *horizontal differential* 1*-forms* are obtained from the horizontal vector fields X_1 and X_2 by the usual musical isomorphism $\flat$: $X_i^\flat = \mathrm{d}x_i$, $i = 1, 2$. For the Reeb vector field we have $T^\flat = \tau$. We use the notation E_0^j, $j = 0, 1, 2, 3$, for the horizontal j-forms: $E_0^0 = \Omega^0$,

$$\begin{aligned} E_0^1 &= \mathrm{span}\{\mathrm{d}x_1, \mathrm{d}x_2\}, \\ E_0^2 &= \mathrm{span}\{\mathrm{d}x_1 \wedge \tau, \mathrm{d}x_2 \wedge \tau\}, \\ E_0^3 &= \mathrm{span}\{\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \tau\}. \end{aligned} \tag{2.12}$$

From the actions of d_c on E_0^* we need the property

$$\mathrm{d}_c^2 = 0 \tag{2.13}$$

and the action $\mathrm{d}_c : E_0^1 \to E_0^2$ that is given, for $\alpha = \alpha_1\mathrm{d}x_1 + \alpha_2\mathrm{d}x_2$, by

$$\mathrm{d}_c\alpha = -P_2(\alpha_1, \alpha_2)\mathrm{d}x_1 \wedge \tau + P_1(\alpha_1, \alpha_2)\mathrm{d}x_2 \wedge \tau. \tag{2.14}$$

The formal L^2 adjoint $\delta_c : E_0^2 \to E_0^1$ of d_c is given for $\alpha = \alpha_{13}\mathrm{d}x_1 \wedge \tau + \alpha_{23}\mathrm{d}x_2 \wedge \tau \in E_0^2$ by

$$\delta_c\alpha = P_1(\alpha_{23}, -\alpha_{13})\mathrm{d}x_1 + P_2(\alpha_{23}, -\alpha_{13})\mathrm{d}x_2. \tag{2.15}$$

We emphasize that the resulting Rumin complex (E_0^*, d_c) is exact, see Theorem 5.8(v) in [9].

The Hodge duality in (Ω^*, d) with respect to the usual scalar product and the volume form $\mathrm{d}V := \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \tau$ is denoted by $*$. Its action on horizontal 1-forms $* : E_0^1 \to E_0^2$ is given by $*(\mathrm{d}x_1) = \mathrm{d}x_2 \wedge \tau$, $*(\mathrm{d}x_2) = -\mathrm{d}x_1 \wedge \tau$. The action of the skew-involution J on horizontal 1-forms, which we again denote by $J : E_0^1 \to E_0^1$, is naturally given by conditions $J(\mathrm{d}x_1) = \mathrm{d}x_2$, $J(\mathrm{d}x_2) = -\mathrm{d}x_1$. Therefore one gets an action $*J : E_0^1 \to E_0^2$ given by $*J(\mathrm{d}x_1) = -\mathrm{d}x_1 \wedge \tau$, $*J(\mathrm{d}x_2) = -\mathrm{d}x_2 \wedge \tau$, whence

$$(*J)(\alpha) = -\alpha \wedge \tau \qquad \text{for every } \alpha \in E_0^1. \tag{2.16}$$

The relation between $\mathrm{d}_c : E_0^1 \to E_0^2$ and $\delta_c : E_0^2 \to E_0^1$ then reads $\mathrm{d}_c = *\delta_c*$.

Note that the operator $\frac{1}{4} J\nabla_{\mathbb{H}}$ appearing in (2.11) is related to the vector field

$$X_u := \frac{1}{4} J\nabla_{\mathbb{H}} u$$

which traditionally goes by the name of the *symplectic gradient* of u. The vector field X_u is uniquely determined by the condition

$$X_u \lrcorner \mathrm{d}\tau = -\mathrm{d}_c u. \tag{2.17}$$

3. The conductivity equation in the Heisenberg group

In this section we introduce the conductivity equation associated to a complex dilatation on the Heisenberg group. This is a linear PDE system which is satisfied by the components of any quasiregular mapping with the given dilatation.

We are motivated by the corresponding theory for planar quasiregular maps [1, Chapter 16]. Let us recall that the hyperbolic disc $(\mathbb{D}, \rho_{\mathbb{D}})$ ($\rho_{\mathbb{D}}$ denotes the hyperbolic metric in $\mathbb{D}$) is isometric to the space

$$S(2) = \{\sigma \in M_{2\times 2}(\mathbb{R}) : \sigma J \sigma = J,\, \mathrm{tr}(\sigma) > 0\}. \tag{3.1}$$

of symmetric positive definite 2×2 matrices Y of determinant one equipped with the distance function ρ_g generated by the Riemannian metric $g = \frac{1}{2}\,\mathrm{tr}(Y^{-1}\, dY)^2$. The isometric identification is given as follows:

$$\mu \to \sigma = \frac{1}{1-|\mu|^2} \begin{pmatrix} |1-\mu|^2 & -2\,\mathrm{Im}\,\mu \\ -2\,\mathrm{Im}\,\mu & |1+\mu|^2 \end{pmatrix} \tag{3.2}$$

or

$$\mu = \frac{A - C - 2\mathbf{i}B}{A + C + 2} \leftarrow \sigma = \begin{pmatrix} A & B \\ B & C \end{pmatrix}. \tag{3.3}$$

We call σ the *conductivity matrix.*

In our derivation of the conductivity equation on the Heisenberg group, we follow the approach indicated in the final section of [3].

Suppose that $F = (f_I, f_3)$ is a K-quasiregular map of a domain in $\mathbb{H}$. In order to derive the conductivity equation, we begin from the Beltrami equation $\overline{Z} f_I = \mu Z f_I$ which we separate into real and imaginary parts by writing $\mu = \alpha + \mathbf{i}\beta$. We obtain

$$(X_1 + \mathbf{i}X_2)(f_1 + \mathbf{i}f_2) = (\alpha + \mathbf{i}\beta)(X_1 - \mathbf{i}X_2)(f_1 + \mathbf{i}f_2),$$

which is equivalent to the pair of equations

$$\begin{aligned} X_1 f_1 - X_2 f_2 &= \alpha(X_1 f_1 + X_2 f_2) - \beta(X_1 f_2 - X_2 f_1), \\ X_2 f_1 + X_1 f_2 &= \beta\,(X_1 f_1 + X_2 f_2) + \alpha\,(X_1 f_2 - X_2 f_1)\,. \end{aligned}$$

Rearranging yields

$$(1-\alpha)X_1f_1 + \beta X_1f_2 = (1+\alpha)X_2f_2 + \beta X_2f_1,$$
$$(1-\alpha)X_1f_2 - \beta X_1f_1 = -(1+\alpha)X_2f_1 + \beta X_2f_2.$$

These equations in turn imply

$$\left((1-\alpha)^2+\beta^2\right)X_1f_2 = 2\beta X_2f_2 + (\beta^2+\alpha^2-1)X_2f_1,$$
$$\left((1-\alpha)^2+\beta^2\right)X_1f_1 = (1-\alpha^2-\beta^2)X_2f_2 + 2\beta X_2f_1,$$

which is equivalent to

$$\frac{(1-\alpha)^2+\beta^2}{1-\alpha^2-\beta^2}X_1f_2 = \frac{2\beta}{1-\alpha^2-\beta^2}X_2f_2 - X_2f_1, \tag{3.4}$$

$$\frac{(1-\alpha)^2+\beta^2}{1-\alpha^2-\beta^2}X_1f_1 = \frac{2\beta}{1-\alpha^2-\beta^2}X_2f_1 + X_2f_2. \tag{3.5}$$

By using the isometry (3.2) we get

$$\begin{aligned} J\sigma\begin{pmatrix}X_1f_1\\X_2f_1\end{pmatrix} &= \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}\frac{(1-\alpha)^2+\beta^2}{1-\alpha^2-\beta^2} & \frac{-2\beta}{1-\alpha^2-\beta^2}\\ \frac{-2\beta}{1-\alpha^2-\beta^2} & \frac{(1+\alpha)^2+\beta^2}{1-\alpha^2-\beta^2}\end{pmatrix}\begin{pmatrix}X_1f_1\\X_2f_1\end{pmatrix}\\ &= \begin{pmatrix}\frac{2\beta}{1-\alpha^2-\beta^2}X_1f_1 - \frac{(1+\alpha)^2+\beta^2}{1-\alpha^2-\beta^2}X_2f_1\\ \frac{(1-\alpha)^2+\beta^2}{1-\alpha^2-\beta^2}X_1f_1 - \frac{2\beta}{1-\alpha^2-\beta^2}X_2f_1\end{pmatrix} = \begin{pmatrix}X_1f_2\\X_2f_2\end{pmatrix}, \end{aligned}$$

where the last identity follows from (3.4) and (3.5). Hence we have the relation

$$\nabla_{\mathbb{H}}f_2 = J\sigma\nabla_{\mathbb{H}}f_1. \tag{3.6}$$

By using (3.1) we obtain

$$\nabla_{\mathbb{H}}f_1 = -J\sigma\nabla_{\mathbb{H}}f_2. \tag{3.7}$$

By combining (3.1), (3.7) and (3.6) we can equally well write

$$\begin{aligned}\sigma\nabla_{\mathbb{H}}f_1 &= -J\nabla_{\mathbb{H}}f_2,\\ \sigma\nabla_{\mathbb{H}}f_2 &= J\nabla_{\mathbb{H}}f_1.\end{aligned} \tag{3.8}$$

Assuming $F \in C^2_{\mathbb{H}}$ we may take the horizontal divergence of the previous equations to get

$$\begin{aligned}\nabla_{\mathbb{H}}\cdot\sigma\nabla_{\mathbb{H}}f_1 &= -\nabla_{\mathbb{H}}\cdot J\nabla_{\mathbb{H}}f_2 = -4Tf_2,\\ \nabla_{\mathbb{H}}\cdot\sigma\nabla_{\mathbb{H}}f_2 &= \nabla_{\mathbb{H}}\cdot J\nabla_{\mathbb{H}}f_1 = 4Tf_1.\end{aligned} \tag{3.9}$$

If the dilatation μ depends only on the first two coordinates of $\mathbb{H}$, $\mu = \mu(z)$, and if $F \in C^3_{\mathbb{H}}(U)$, then we can apply the operator $\sigma\nabla_{\mathbb{H}}$ to find

$$\sigma\nabla_{\mathbb{H}}(\nabla_{\mathbb{H}}\cdot\sigma\nabla_{\mathbb{H}})f_1 = -\sigma\nabla_{\mathbb{H}}(4Tf_2) = -4T(\sigma\nabla_{\mathbb{H}}f_2) = -4T(J\nabla_{\mathbb{H}}f_1)$$

and similarly for f_2. Applying the same argument starting from the second Beltrami equation $\overline{Z}f_{II} = \mu Zf_{II}$ yields a similar conclusion for f_3 and f_4. We proved

Proposition 3.1. *Let $F = (f_1, f_2, f_3)$ be a $C^3_{\mathbb{H}}$ quasiregular map of a domain $U \subset \mathbb{H}$, whose complex dilatation μ depends only on the z-coordinate. Define $f_4 = |f_I|^2 = f_1^2 + f_2^2$. Then*

$$\left(\sigma\nabla_{\mathbb{H}}(\nabla_{\mathbb{H}}\cdot\sigma\nabla_{\mathbb{H}}) + 4J\nabla_{\mathbb{H}}T\right)f_j = 0, \qquad j = 1,2,3,4, \tag{3.10}$$

where σ is defined in terms of μ by (3.2).

Equation (3.10) is called the *conductivity equation* associated to the matrix σ. Note that it is **third-order** in horizontal partial derivatives. As we will see shortly, (3.10) can be stated using the horizontal curl operator. If $F \in C^4_{\mathbb{H}}(U)$ we can take the horizontal divergence of the expression on the left hand side in (3.10) to derive the following fourth-order equation:

$$\left((\nabla_{\mathbb{H}} \cdot \sigma \nabla_{\mathbb{H}})^2 + 16T^2 \right) f_j = 0, \qquad j = 1,2,3,4. \tag{3.11}$$

For instance, 1-quasiregular maps of $\mathbb{H}$ have components which satisfy (2.2) with $\mu = 0$ or equivalently, satisfy (3.9) with $\sigma = \mathrm{Id}_2$. Such components satisfy the fourth order equation

$$\left(\mathcal{L}^2_{\mathbb{H}} + 16T^2 \right) f_j = 0, \qquad j = 1,2,3,4, \tag{3.12}$$

where $\mathcal{L}_{\mathbb{H}} = \nabla_{\mathbb{H}} \cdot \nabla_{\mathbb{H}} = X_1^2 + X_2^2$ denotes the Kohn sub-Laplacian on $\mathbb{H}$. Note that Liouville–Gehring–Reshetnyak rigidity holds for 1-quasiregular maps on the Heisenberg group; every such map defined on a domain $U \subset \mathbb{H}$ is the restriction to U of a Möbius transformation of the Heisenberg group (which in turn corresponds to the action of an element of $SU(2,1)$ on the compactified Heisenberg group). In particular, such maps are smooth. This was shown for C^4 maps by Korányi and Reimann [11] and later by Capogna [4] without the regularity assumption.

Remark 3.2. We note also that the operator on the left hand side of the equation (3.11) factors into a pair of second order operators

$$(\nabla_{\mathbb{H}} \cdot \sigma \nabla_{\mathbb{H}})^2 + 16T^2 = 16 \Box_{\sigma,1} \Box_{\sigma,-1},$$

where

$$\Box_{\sigma,\alpha} = -\frac{1}{4} \nabla_{\mathbb{H}} \cdot \sigma \nabla_{\mathbb{H}} + \mathbf{i} \alpha T.$$

Solvability of the operator $\Box_{S,\alpha}$ for complex symplectic matrices S on the Heisenberg group $\mathbb{H}_n$ was studied by Müller, Peloso and Ricci in [14]. The operators $\Box_{\sigma,1}$ and $\Box_{\sigma,-1}$ are both not solvable. The case $\sigma = \mathrm{Id}_2$ coincides with Folland-Stein operators, see [7] or [17, Chapter XIII.2.2].

Now let us assume that μ depends on all of the coordinates of $\mathbb{H}$, i.e., $\mu = \mu(z, x_3)$. Then $T\mu$, hence also $T\sigma$, will no longer vanish. In this case the components of F will satisfy a more complicated conductivity equation involving additional terms.

We return to (3.9) and again apply $\sigma \nabla_{\mathbb{H}}$ to the first equation in (3.9) to obtain

$$\begin{aligned} \sigma \nabla_{\mathbb{H}} \big(\nabla_{\mathbb{H}} \cdot \sigma \nabla_{\mathbb{H}} f_1 \big) = \sigma \nabla_{\mathbb{H}} \big(-4T f_2 \big) &= -4T \big(\sigma \nabla_{\mathbb{H}} f_2 \big) + 4(T\sigma) \nabla_{\mathbb{H}} f_2 \\ &= -4TJ \nabla_{\mathbb{H}} f_1 + 4(T\sigma) J \sigma \nabla_{\mathbb{H}} f_1 \\ &= -4JT \nabla_{\mathbb{H}} f_1 - 4\sigma J (T\sigma) \nabla_{\mathbb{H}} f_1. \end{aligned} \tag{3.13}$$

In the last line we used the identity $(T\sigma) J \sigma + \sigma J (T\sigma) = 0$ which comes from applying the operator T to the identity

$$\sigma J \sigma = J. \tag{3.14}$$

Rearranging (3.13) and using (3.14) yields

$$J \sigma T \nabla_{\mathbb{H}} f_1 + J (T\sigma) \nabla_{\mathbb{H}} f_1 + \tfrac{1}{4} \nabla_{\mathbb{H}} \big(\nabla_{\mathbb{H}} \cdot \sigma \nabla_{\mathbb{H}} f_1 \big) = 0$$

or

$$JT \big(J \sigma \nabla_{\mathbb{H}} f_1 \big) + \tfrac{1}{4} J \nabla_{\mathbb{H}} \big(\nabla_{\mathbb{H}} \cdot \sigma \nabla_{\mathbb{H}} f_1 \big) = 0.$$

In view of (2.10) and repeating the argument for f_2, f_3, f_4 we arrive at the following conclusion.

Theorem 3.3. *Let $F = (f_1, f_2, f_3)$ be a $C^3_{\mathbb{H}}$ quasiregular map of a domain $U \subset \mathbb{H}$ with complex dilatation $\mu = \mu(z, x_3)$. Define $f_4 = |f_I|^2 = f_1^2 + f_2^2$. Then*

$$\nabla_{\mathbb{H}} \times \big(J\sigma\nabla_{\mathbb{H}} f_j\big) = 0, \qquad j = 1, 2, 3, 4, \tag{3.15}$$

where σ is defined in terms of μ by (3.2).

The vector field

$$X^\sigma_u := \frac{1}{4} J\sigma\nabla_{\mathbb{H}} u$$

is the unique vector field satisfying condition

$$(\sigma X^\sigma_u) \lrcorner \mathrm{d}\tau = -\mathrm{d}_c u$$

for $u \in W^{1,p}(U)$, $U \in \mathbb{H}$. The case $\sigma = \mathrm{Id}_2$ corresponds the condition (2.17) for the symplectic gradient. It is tempting to call X^σ_u the *σ-symplectic gradient* of u. We note that due to (3.14), $J\sigma$ is also a skew-involution, that is, $(J\sigma)^2 = -\mathrm{Id}_2$ holds.

Note that when the complex dilatation μ satisfies the bound (2.3), then σ verifies the ellipticity bounds

$$K^{-1}|\xi|^2 \le \langle\sigma\xi, \xi\rangle_p \le K|\xi|^2, \quad \text{for every } \xi \in \operatorname{span}\{X_1, X_2\}(p),$$

where K and k are related by (2.4). Here $\langle\cdot,\cdot\rangle_p$ is the standard inner product in the horizontal tangent plane $\operatorname{span}\{X_1, X_2\}(p)$, $p \in \mathbb{H}$.

One can formulate the conductivity equation (3.15) also in terms of horizontal differential forms introduced in section 2.2. We get

$$\mathrm{d}_c(J\sigma \mathrm{d}_c u) = 0, \tag{3.16}$$

where the function $u \in C^3_{\mathbb{H}}(U)$ is a solution of (3.15) and the conductivity σ is as in Theorem 3.3. By utilizing the adjoint operator $\delta_c : E^2_0 \to E^1_0$ in (2.15) we get an equivalent equation

$$\delta_c(\sigma \mathrm{d}_c u \wedge \tau) = 0, \tag{3.17}$$

since

$$\mathrm{d}_c(J\sigma \mathrm{d}_c u) = *\delta_c(*J)(\sigma \mathrm{d}_c u) = - * \delta_c(\sigma \mathrm{d}_c u \wedge \tau),$$

where equation (2.16) is also used.

Equation (3.17) allows us to formulate a weak version of the conductivity equation.

Definition 3.4. A function $u \in W^{1,p}_{\mathbb{H},loc}(U)$ is a *weak solution of the conductivity equation* (3.17) if for every $\varphi = \varphi_1 \mathrm{d}x_1 + \varphi_2 \mathrm{d}x_2 \in E^1_0$ with $\varphi_i \in C^\infty_0(U)$

$$\int_U \langle\sigma \mathrm{d}_c u \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p = 0 \tag{3.18}$$

holds. Above $\langle\cdot,\cdot\rangle.$ denotes the natural inner product in the space of horizontal two forms E^2_0. The underlying measure is the three dimensional Lebesgue measure which agrees with Haar measure in the group $\mathbb{H}$.

We can now formulate the weak version of Theorem 3.3.

Theorem 3.5. *Let $F = (f_1, f_2, f_3)$ be a $W^{1,4}_{\mathbb{H},loc}(U)$ quasiregular map of a domain $U \subset \mathbb{H}$ with complex dilatation $\mu = \mu(z, x_3)$. Define $f_4 = |f_I|^2 = f_1^2 + f_2^2$. Then, for each $j =$ 1, 2, 3, 4, the function $u = f_j$ is a weak solution of the conductivity equation.*

Suppose now that $u \in W^{1,4}_{\mathbb{H}}(U)$ is a weak solution of the conductivity equation (3.18). Mimicking the electrostatistic analogy in the plane, we define a *div-curl couple* $\mathcal{F} = [\mathbf{E}, \mathbf{B}]$ by setting

$$\begin{aligned} \mathbf{E} &= \mathrm{d}_c u \in E_0^1, \\ \mathbf{B} &= \sigma \mathrm{d}_c u \wedge \tau \in E_0^2. \end{aligned} \tag{3.19}$$

Since $\mathrm{d}_c^2 = 0$ holds in the horizontal complex (E_0^*, d_c) we immediately get that $\mathrm{d}_c \mathbf{E} = 0$ holds and $\mathbf{E}$ is a curl free vector field. For a general horizontal vector field V one can define (see [9]) an intrinsic *divergence operator* $\mathrm{Div}_{\mathbb{H}}$ by setting

$$\mathrm{Div}_{\mathbb{H}} V := *\delta_c V^\flat,$$

where $\delta_c : E_0^1 \to E_0^0$ acts on 1-forms via $\delta_c = - * \mathrm{d}_c *$. Then the vector field $\mathbf{B}$ plays the role of a divergence free vector field, since $\delta_c \mathbf{B} = 0$ holds.

One can further introduce quantities

$$|\mathcal{F}|^2 = |\mathbf{E}|^2 + |\mathbf{B}|^2 = |\mathbf{E} \wedge \tau|^2 + |\mathbf{B}|^2$$

and

$$J_{\mathcal{F}} = \langle \mathbf{E} \wedge \tau, \mathbf{B} \rangle$$

and call a div-curl couple $\mathcal{F} = [\mathbf{B}, \mathbf{E}]$ that satisfies the distortion equality

$$|\mathcal{F}|^2 \le \left(K + \frac{1}{K} \right) J_{\mathcal{F}}$$

a *K-quasiconformal field* in the Heisenberg group. This terminology is analogous to the plane case [1, Chapter 16.1.6] where it arises in connection with the Hodge $*$ method. In classical electrodynamics $\mathbf{E}$ and $J\mathbf{B}$ give rise to Faraday's form as introduced in [9]. We will return to this facinating connection elsewhere.

Remark 3.6. In the planar case, both divergence $\nabla \cdot V = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2}$ and curl $\nabla \times V = \frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1}$ are scalar functions acting on a vector field $V = (V_1, V_2)$. They are related by the Hodge $*$ operator which acts on vector fields by multiplication by the matrix J. Thus, on a planar domain U the conductivity equation can be formulated either in divergence form:

$$\nabla \cdot (\sigma(z) \nabla u) = 0 \tag{3.20}$$

or in terms of the curl:

$$\nabla \times (J\sigma(z) \nabla u) = 0. \tag{3.21}$$

Due to the Poincaré lemma, if $u \in W^{1,1}_{loc}(U)$ is a solution to (3.21) then there exists $v \in W^{1,1}_{loc}(U)$ with $\nabla v = J\sigma(z)\nabla u$. In this case $\mathbf{B} = \sigma \nabla u$ is divergence free (see (3.20)) while $\mathbf{E} = \nabla u$ is curl free. The pair $[\mathbf{B}, \mathbf{E}]$ is termed a *div-curl couple*.

In the Heisenberg case this duality between divergence and curl is no longer present in the same form. The horizontal divergence is a scalar valued operator, while the horizontal curl is a vector valued operator. We have formulated the Heisenberg conductivity equation first in terms of the horizontal curl operator in (3.15), then equivalently in terms of horizontal differential forms and the horizontal exterior derivative acting on 1-forms in (3.16), and finally in terms of the horizontal adjoint operator acting on horizontal 2-forms in (3.17). It is the adjoint operator that can be related to the divergence operator. It is in this sense clear what the notion of div-curl couple means in this setting.

Additional tools are available in the planar case, namely, Stoilow factorization and the measurable Riemann mapping theorem. Every solution to a conductivity equation with strongly elliptic conductivity matrix can be written as the composition of a harmonic function and a quasiconformal mapping, see Theorem 16.1.4 in [1] for details. There is no obvious parallel to this result in the Heisenberg case.

4. Conductivity solutions generate quasiregular maps

We now reverse the process described in the previous section. Starting from a solution to the conductivity equation (3.15), subject to a compatibility assumption (4.5), we generate a quasiregular map between domains in $\mathbb{H}$.

Let us fix a measurable conductivity matrix function $\sigma : U \to S(2)$ defined on a simply connected domain $U \subset \mathbb{H}$ and satisfying the ellipticity bound

$$\frac{1}{K}|\xi|^2 \leq \langle \sigma\xi, \xi\rangle \leq K|\xi|^2, \quad \xi \in \operatorname{span}\{X_1, X_2\} \tag{4.1}$$

for some $K \geq 1$. By considering the action of σ on an orthonormal eigenbasis it is easy to see that the two-sided estimate in (4.1) can be written as a single inequality:

$$|\xi|^2 + |\sigma\xi|^2 \leq \left(K + \frac{1}{K}\right)\langle \sigma\xi, \xi\rangle, \quad \xi \in \operatorname{span}\{X_1, X_2\}. \tag{4.2}$$

Let $u = f \in C^3_{\mathbb{H}}(U)$ be a solution to the conductivity equation

$$\nabla_{\mathbb{H}} \times \big(J\sigma\nabla_{\mathbb{H}} u\big) = 0. \tag{4.3}$$

By Theorem 2.4(b) there exists $g \in C^3_{\mathbb{H}}(U)$ so that $J\sigma\nabla_{\mathbb{H}} f = \nabla_{\mathbb{H}} g$ in U. Then, since

$$J\sigma\nabla_{\mathbb{H}} g = (J\sigma)^2\nabla_{\mathbb{H}} f = -\nabla_{\mathbb{H}} f$$

holds, $u = g$ is also a solution to (4.3) and (3.8) holds for the pair (f, g), i.e.,

$$\begin{aligned} \sigma\nabla_{\mathbb{H}} f &= -J\nabla_{\mathbb{H}} g, \\ \sigma\nabla_{\mathbb{H}} g &= J\nabla_{\mathbb{H}} f. \end{aligned} \tag{4.4}$$

Theorem 4.1. *Let σ be a conductivity matrix on a simply connected domain $U \subset \mathbb{H}$ and assume that σ satisfies the ellipticity bound (4.1). Let $f \in C^3_{\mathbb{H}}(U)$ solve (4.3) and let g be the conjugate function as described above. If*

$$u := f^2 + g^2 \text{ satisfies the conductivity equation (4.3),} \tag{4.5}$$

then there exists $h \in C^3_{\mathbb{H}}(U)$ so that $F = (f, g, h)$ is a K-quasiregular map of U.

Proof. The first step of the proof is to construct the third coordinate function h and verify the contact condition

$$\nabla_{\mathbb{H}} h + 2f\nabla_{\mathbb{H}} g - 2g\nabla_{\mathbb{H}} f = 0. \tag{4.6}$$

To do this, we will take advantage of Theorem 2.4(b). It suffices to verify that the vector field $f\nabla_{\mathbb{H}} g - g\nabla_{\mathbb{H}} f$ has vanishing horizontal curl.

Since $u = f^2 + g^2$ also satisfies the conductivity equation we know that

$$\nabla_{\mathbb{H}} \times (fJ\sigma\nabla_{\mathbb{H}} f + gJ\sigma\nabla_{\mathbb{H}} g) = 0.$$

Using system (4.4) on the left hand side of the above equation yields

$$\nabla_{\mathbb{H}} \times (f\nabla_{\mathbb{H}} g - g\nabla_{\mathbb{H}} f) = 0$$

as desired. By Theorem 2.4(b) there exists h so that (4.6) is satisfied. In other words, the map $F = (f, g, h)$ is a generalized contact map. To see that $h \in C^3_{\mathbb{H}}$ it suffices to observe that $f\nabla_{\mathbb{H}} g - g\nabla_{\mathbb{H}} f \in C^2_{\mathbb{H}}(U)$. Let us also remark in passing that the pair of functions h and $u = f^2 + g^2$ satisfies the system (4.4) for the same conductivity matrix σ.

It remains to verify the dilatation bound (2.1). Using the identity

$$||A||^2 = \tfrac{1}{2}||A||^2_{HS} + \sqrt{\tfrac{1}{4}||A||^4_{HS} - \det{}^2 A}$$

valid for 2×2 matrices A, where $||A||$ denotes the operator norm of A and $||A||_{HS}$ denotes the Hilbert–Schmidt norm, we observe that it suffices to show

$$||D_H F||^2_{HS} \le \left(K + \frac{1}{K}\right) \det D_H F \qquad \text{a.e. in } U. \tag{4.7}$$

Since $||D_H F||^2_{HS} = |\nabla_{\mathbb{H}} f|^2 + |\nabla_{\mathbb{H}} g|^2$ and $\det D_H F = \langle \nabla_{\mathbb{H}} g, J\nabla_{\mathbb{H}} f\rangle$, (4.7) can be rewritten

$$|\nabla_{\mathbb{H}} f|^2 + |\nabla_{\mathbb{H}} g|^2 \le \left(K + \frac{1}{K}\right) \langle \nabla_{\mathbb{H}} g, J\nabla_{\mathbb{H}} f\rangle \qquad \text{a.e. in } U. \tag{4.8}$$

We use the system (4.4) to rewrite (4.8) in terms of a single component f as follows:

$$|\nabla_{\mathbb{H}} f|^2 + |\sigma\nabla_{\mathbb{H}} f|^2 \le \left(K + \frac{1}{K}\right) \langle \nabla_{\mathbb{H}} f, \sigma\nabla_{\mathbb{H}} f\rangle \qquad \text{a.e. in } U.$$

This is precisely (4.2). Hence F is K-quasiregular. □

The above theorem also has a corresponding weak formulation.

Theorem 4.2. *Let σ be a conductivity matrix on a simply connected domain $U \subset \mathbb{H}$ and assume that σ satisfies the ellipticity bound* (4.1)*. Let $f \in W^{1,4}_{\mathbb{H},loc}(U)$ be a continuous solution to* (3.18) *and let $g \in W^{1,4}_{\mathbb{H},loc}(U)$ be its conjugate function. If $u := f^2 + g^2$ satisfies* (3.18)*, then there exists $h \in W^{1,4}_{\mathbb{H},loc}(U)$ so that $F = (f, g, h)$ is a K-quasiregular map of U.*

Proof. Below we write down the places where the weak formulation is used. Quasiregularity for F follows as in the previous theorem.

Since f solves (3.18) we also find $g \in W^{1,4}_{\mathbb{H},loc}(U)$ so that $J\sigma d_c f = d_c g$ in the sense of distributions and furthermore, both $J\mathrm{d}_c f = \sigma J\sigma \mathrm{d}_c f = \sigma \mathrm{d}_c g$ and $-\sigma \mathrm{d}_c f = J\mathrm{d}_c g$ hold.

For every $\varphi = \varphi_1 \mathrm{d}x_1 + \varphi_2 \mathrm{d}x_2 \in E^1_0$ with $\varphi_i \in C^\infty_0(U)$ we now get

$$\begin{aligned}\int_U \langle \sigma \mathrm{d}_c g \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p &= \int_U \langle (J\mathrm{d}_c f) \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p \\ &= -\int_U \langle (*J)(J\mathrm{d}_c f) \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p = \int_U \langle *\mathrm{d}_c f, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p \\ &= -\int_U \langle *\mathrm{d}_c f, *\delta_c * \varphi\rangle_p \,\mathrm{d}p = \int_U \langle \mathrm{d}_c f, \delta_c * \varphi\rangle_p \,\mathrm{d}p = 0.\end{aligned}$$

Above in the second line we used (2.16). It hence follows that g is a solution to (3.18).

To find the third function $h \in W^{1,4}_{\mathbb{H},loc}(U)$ it is now enough to show that

$$\int_U \langle (Jf\mathrm{d}_c g - Jg\mathrm{d}_c f) \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p = 0, \tag{4.9}$$

since for $\alpha \in E_0^1$ the condition $\mathrm{d}_c\alpha = 0$ is equivalent to the condition $\delta_c(J\alpha \wedge \tau) = 0$. From the assumption that $u = f^2 + g^2$ solves (3.18) we get

$$\begin{aligned}\int_U \langle (Jf\mathrm{d}_c g - Jg\mathrm{d}_c f) \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p &= \int_U \langle (fJ\mathrm{d}_c g - gJ\mathrm{d}_c f) \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p \\ &= -\int_U \langle \sigma(f\mathrm{d}_c f + g\mathrm{d}_c g) \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p \\ &= -\frac{1}{2}\int_U \langle \sigma\mathrm{d}_c u \wedge \tau, \mathrm{d}_c\varphi\rangle_p \,\mathrm{d}p = 0.\end{aligned}$$

We obtain functions f, g and h so that the triple $F = (f, g, h)$ is a generalized contact map of $\mathbb{H}$ which satisfies the distortion inequality (2.1). By Theorem 2.2, F is continuous. Hence F is a quasiregular map. As previously mentioned, the K-quasiregularity of F follows exactly as in the proof of Theorem 4.1. □

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University of Cambridge, Wilberforce Road, Cambridge,CB3 0WB
E-mail address: aai22@cam.ac.uk

Aalto University, P.O. Box 11100, 00076 Aalto, Finland
E-mail address: kirsi.peltonen@aalto.fi

University of Illinois at Urbana-Champaign 1409 West Green Street Urbana, IL 61801, USA
E-mail address: tyson@math.uiuc.edu