The higher integrability of the gradient for systems of porous medium type

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The porous medium equation

Model case:

- For *non-negative* solutions $u : \Omega \times (0, T) \to [0, \infty)$, the porous medium equation reads

$$\partial_t u - \Delta u^m = 0, \qquad \text{in } \Omega_T := \Omega \times (0, T),$$

for $\boldsymbol{m > 1}$ and $\Omega \subset \mathbb{R}^n$.

- *Vector-valued case:* The porous medium system for $u : \Omega_T \to \mathbb{R}^N$ reads

$$\partial_t u - \Delta\big(|u|^{m-1} u\big) = 0 \qquad \text{in } \Omega_T.$$

For the power of a vector $u \in \mathbb{R}^N$, we abbreviate

$$\boldsymbol{u}^m := |u|^{m-1} u.$$

Some properties of the PME

$$\partial_t u - \Delta \boldsymbol{u}^m = 0 \qquad \text{in } \Omega_T.$$

In this talk, we restrict ourselves to the **degenerate case $\boldsymbol{m} > \mathbf{1}$**.

- *Degeneracy:*
  The modulus of ellipticity of the diffusion part degenerates if $|u|$ is small;
- *Anisotropic scaling behaviour:*
  If u is a solution, cu with $c \in \mathbb{R}$ is in general no solution;
- If u is a solution, $u + c$ with $c \in \mathbb{R}$ is no solution.

The general case

$$\partial_t u - \mathrm{div} A(x, t, u, D\boldsymbol{u}^m) = \mathrm{div} F \qquad \text{in } \Omega_T, \tag{PMS}$$

where

- $F \in L^{2+\delta}(\Omega_T, \mathbb{R}^{Nn})$;
- $A : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is a Carathéodory function with

$$\begin{cases} A(x, t, u, \xi) \cdot \xi \geq \nu |\xi|^2, \\ |A(x, t, u, \xi)| \leq L|\xi|, \end{cases}$$

for $0 < \nu \leq L$, a.e. $(x, t) \in \Omega_T$ and all $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$.

The model case is $A(x, t, u, \xi) = A(\xi) = \xi$.

Definition

A *weak solution* to the porous medium type system (PMS) is a measurable map $u : \Omega_T \to \mathbb{R}^N$ with

$$\begin{cases} u \in C^0\big((0,T); L^{m+1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\big), \\ \boldsymbol{u}^m \in L^2_{\mathrm{loc}}\big(0,T; W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\big), \end{cases}$$

so that for any $\varphi \in C_0^\infty(\Omega_T,\mathbb{R}^N)$, we have

$$\iint_{\Omega_T} \big[u\cdot\partial_t\varphi - \mathbf{A}(x,t,u,D\boldsymbol{u}^m)\cdot D\varphi\big]\,dx\,dt = \iint_{\Omega_T} F\cdot D\varphi\,dx\,dt.$$

Topic of the talk:

Is there an $\varepsilon > 0$ with $|D\boldsymbol{u}^m| \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega_T)$?

History of the problem – The elliptic case

- Gehring (1973): Higher integrability of the Jacobian of a quasi-conformal mapping
- Elcrat & Meyers (1975): Higher integrability of the gradient for solutions of nonlinear elliptic systems of p-Laplace type
- Giaquinta & Modica (1986): Higher integrability of the gradient for minimizers of variational integrals

Since then, there have been countless generalizations and applications to the regularity theory of elliptic systems.

The key step in the proof is a reverse Hölder inequality of the type

$$\fint_{B_{R/2}} |Du|^p \, dx \le c \left(\fint_{B_R} |Du|^{\theta p} \, dx \right)^{\frac{1}{\theta}} + c \fint_{B_R} |F|^p \, dx$$

for $\theta \in (0,1)$ and $F \in L^{p+\varepsilon}(\Omega)$, which makes it possible to apply Gehring's lemma.

History of the problem – the parabolic case

- Giaquinta & Struwe (1982): Higher integrability for quasilinear parabolic systems
- Kinnunen & Lewis (2000): Parabolic p-Laplace systems, $p > \frac{2n}{n+2}$.

For degenerate parabolic systems of p-Laplace-type, a serious problem arises since

$$\partial_t u - \operatorname{div}(|Du|^{p-2} Du) = \operatorname{div}(|F|^{p-2} F)$$

has a different scaling behaviour in the time- and the diffusion part. Therefore, one does not obtain a homogeneous reverse Hölder inequality, which is necessary for the application of Gehring's lemma.

Intrinsic geometry by DiBenedetto

Kinnunen & Lewis overcame this problem by using the idea of intrinsic geometry by DiBenedetto. The idea is to consider parabolic cylinders adapted to the modulus of ellipticity $|Du|^{p-2}$ of the system

$$\partial_t u - \operatorname{div}\big(|Du|^{p-2}Du\big) = 0.$$

Rough Heuristics:

If $|Du| \approx \lambda$, then the p-Laplace system scales like

$$\partial_t u - \lambda^{p-2}\Delta u = 0,$$

and the natural cylinders to consider this problem on have the form

$$Q_\varrho^{(\lambda)}(x_o, t_o) := B_\varrho(x_o) \times (t_o - \lambda^{2-p}\varrho^2, t_o + \lambda^{2-p}\varrho^2).$$

Reverse Hölder inequality on intrinsic cylinders

Lemma (Kinnunen & Lewis, 2000)

Suppose that $p \geq 2$, and that u is a weak solution of the system

$$\partial_t u - \operatorname{div}\big(|Du|^{p-2}Du\big) = \operatorname{div}(|F|^{p-2}F) \qquad \text{in } \Omega_T,$$

and that on $Q_{20\varrho}^{(\lambda)} \subset \Omega_T$, we have an intrinsic coupling of the form

$$\lambda^p \leq c \fint\!\!\!\fint_{Q_\varrho^{(\lambda)}} \big(|Du|^p + |F|^p\big)\,dxdt \leq \tilde{c} \fint\!\!\!\fint_{Q_{20\varrho}^{(\lambda)}} \big(|Du|^p + |F|^p\big)\,dxdt \leq \hat{c}\lambda^p$$

Then we have the reverse Hölder inequality

$$\fint\!\!\!\fint_{Q_\varrho^{(\lambda)}} |Du|^p\,dxdt \leq c\left(\fint\!\!\!\fint_{Q_{4\varrho}^{(\lambda)}} |Du|^{\theta p}\,dx\right)^{\frac{1}{\theta}} + \fint\!\!\!\fint_{Q_{4\varrho}^{(\lambda)}} |F|^p\,dxdt$$

for some $\theta \in (0,1)$.

Stopping time argument

The intrinsic coupling can be established by a stopping time argument:

$$\text{For } z_o \in E(\lambda) := \{z \in \Omega_T \,:\, |Du(z)| > \lambda\},$$

we have, on the one hand,

$$\lim_{\varrho \downarrow 0} \iint\!\!\!\!\!\!\!-\,_{Q_\varrho^{(\lambda)}(z_o)} \big(|Du|^p + |F|^p\big)\, dx\, dt > \lambda^p,$$

and on the other hand,

$$\iint\!\!\!\!\!\!\!-\,_{Q_\varrho^{(\lambda)}(z_o)} \big(|Du|^p + |F|^p\big)\, dx\, dt \le c\lambda^{p-2}\varrho^{-n-2}\big(\|Du\|_{L^p}^p + \|F\|_{L^p}^p\big) \le \lambda^p$$

for $\varrho \gg 1$, provided $\lambda^2 \gg \|Du\|_{L^p}^p + \|F\|_{L^p}^p$. Therefore, there exists a maximal radius ϱ_{z_o} with

$$\iint\!\!\!\!\!\!\!-\,_{Q_{\varrho_{z_o}}^{(\lambda)}(z_o)} \big(|Du|^p + |F|^p\big)\, dx\, dt = \lambda^p.$$

A **Vitali type covering argument** yields a countable covering of the super-level sets $E(\lambda)$ by cylinders with an intrinsic coupling condition.

The reverse Hölder inequality on these cylinders can be used to derive a higher integrability estimate of the form

$$\iint_{Q_R} |Du|^{p+\varepsilon}\, dxdt \leq c \left(\iint_{Q_{2R}} \left(|Du|^p + |F|^p\right) dxdt\right)^{1+\frac{\varepsilon d}{p}} + c \iint_{Q_{2R}} |F|^{p+\varepsilon}\, dxdt$$

with $d := \frac{p}{2}$ for $p \geq 2$ and $d := \frac{2p}{p(n+2)-2n}$ for $\frac{2n}{n+2} < p < 2$.

The porous medium equation

The case of the porous medium equation

$$\partial_t u - m\,\mathrm{div}(u^{m-1}Du) = \partial_t u - \Delta u^m = 0$$

stayed open for a long time.

Main additional problem:

The modulus of ellipticity mu^{m-1} depends on u, but we want to prove estimates for Du^m.

The equation forces us to work with intrinsic cylinders $Q_\varrho^{(\theta)}$ depending on the parameter

$$\theta^{2m} \approx \iint_{Q_\varrho^{(\theta)}} \frac{|u|^{2m}}{\varrho^2}\,dxdt,$$

but for proving estimates for $|D\boldsymbol{u}^m|$, the relevant quantity is

$$\lambda^{2m} \approx \iint_{Q_\varrho^{(\theta)}} |D\boldsymbol{u}^m|^2\,dxdt.$$

Intrinsic geometry adapted to the PME

The modulus of ellipticity $|u|^{m-1}$ suggests to work with cylinders of the form

$$B_\varrho(x_o) \times (t_o - \vartheta\varrho^2, t_o + \vartheta\varrho^2) \qquad \text{with } \vartheta \approx |u|^{1-m}.$$

But we have to work with a parameter θ^m of the same dimension as $|D\mathbf{u}^m|$, i.e.

$$\theta^m \approx \frac{|u|^m}{\varrho} \approx \frac{\vartheta^{\frac{m}{1-m}}}{\varrho}.$$

Therefore, we have to choose $\vartheta = \varrho^{\frac{1-m}{m}}\theta^{1-m}$ above, which leads to the cylinders

$$Q_\varrho^{(\theta)}(x_o, t_o) := B_\varrho(x_o) \times \left(t_o - \theta^{1-m}\varrho^{\frac{m+1}{m}}, t_o + \theta^{1-m}\varrho^{\frac{m+1}{m}}\right).$$

In particular, the natural parabolic dimension for the problem is $d := n + \frac{m+1}{m}$.

The construction by Gianazza & Schwarzacher

The problem of higher integrability was solved by Gianazza & Schwarzacher in the case of non-negative solutions.
Their idea is to construct a family of **sub-intrinsic cylinders** $Q_\varrho^{(\theta_{z_o;\varrho})}(z_o)$, i.e.

$$\iint_{Q_\varrho^{(\theta_{z_o;\varrho})}(z_o)} \frac{|u|^{2m}}{\varrho^2}\,dxdt \le \theta_{z_o;\varrho}^{2m}\,.$$

These cylinders satisfy (among other properties)

- $Q_\varrho^{(\theta_{z_o;\varrho})}(z_o) \subset Q_r^{(\theta_{z_o;r})}(z_o)$ for $\varrho < r$;
- $\varrho \mapsto \theta_{z_o;\varrho}$ is continuous for each $z_o \in \Omega_T$;
- a Vitali type covering property.

The result by Gianazza & Schwarzacher reads as follows (note that they start with the regularity $Du^{\frac{m+1}{2}} \in L^2(\Omega_T)$ for the weak solution and are insterested in higher integrability for $Du^{\frac{m+1}{2}}$):

Theorem (Gianazza & Schwarzacher, 2016)

Consider a **non-negative** *weak solution $u \geq 0$ of the porous medium type equation*

$$\partial_t u - \operatorname{div} A(x, t, u, Du^m) = f \qquad \text{in } \Omega_T$$

with $m > 1$ and $f \in L^{\frac{m+1}{m-1}}(\Omega_T, \mathbb{R}_{\geq 0})$. Then, its gradient satisfies

$$\left| Du^{\frac{m+1}{2}} \right| \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega_T) \qquad \text{for some } \varepsilon > 0.$$

Their proof relies on an **expansion of positivity** argument and is therefore limited to non-negative solutions.

The case of systems

The first generalization to the vector-valued case is the following:

Theorem (Bögelein, Duzaar, Korte, S., 2018)

Any weak solution $u : \Omega_T \to \mathbb{R}^N$ of the porous medium type system

$$\partial_t u - \mathrm{div} A(x, t, u, D\boldsymbol{u}^m) = \mathrm{div} F \qquad \text{in } \Omega_T$$

with $m > 1$ and $F \in L^\sigma(\Omega_T, \mathbb{R}^{Nn})$ for $\sigma > 2$ satisfies

$$\left|D\boldsymbol{u}^m\right| \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega_T) \qquad \text{for some } \varepsilon > 0.$$

The proof relies on the construction of intrinsic cylinders by Gianazza & Schwarzacher, but avoids the argument of expansion of positivity.

In what follows, we will explain some of the key ideas of the proof.

Energy bounds

Testing with $(\boldsymbol{u}^m - \boldsymbol{a}^m)\varphi$, where $a \in \mathbb{R}^N$, yields an energy bound of the type

$$\sup_{t\in\Lambda_r^{(\theta)}} \fint_{B_r} \theta^{m-1} \frac{\left|\boldsymbol{u}^{\frac{m+1}{2}}(\cdot,t) - \boldsymbol{a}^{\frac{m+1}{2}}\right|^2}{\varrho^{\frac{m+1}{m}}}\,dx + \iint_{Q_r^{(\theta)}} \left|D\boldsymbol{u}^m\right|^2 dx\,dt$$
$$\le c \iint_{Q_\varrho^{(\theta)}} \left[\frac{\left|\boldsymbol{u}^m - \boldsymbol{a}^m\right|^2}{(\varrho - r)^2} + \theta^{m-1}\frac{\left|\boldsymbol{u}^{\frac{m+1}{2}}(\cdot,t) - \boldsymbol{a}^{\frac{m+1}{2}}\right|^2}{\varrho^{\frac{m+1}{m}} - r^{\frac{m+1}{m}}}\right] dx\,dt$$
$$+ c \iint_{Q_\varrho^{(\theta)}} |F|^2\,dx\,dt,$$

on any cylinder of the form

$$Q_\varrho^{(\theta)} := B_\varrho(x_o) \times \Lambda_\varrho^{(\theta)} := B_\varrho(x_o) \times \left(t_o - \theta^{1-m}\varrho^{\frac{m+1}{m}}, t_o + \theta^{1-m}\varrho^{\frac{m+1}{m}}\right)$$

for $\theta > 0$ and $0 < r < \varrho \le 1$.

Sobolev-Poincaré inequality

On any *sub-intrinsic* cylinder $Q_\varrho^{(\theta)}$, i.e. under the assumption

$$\iint_{Q_\varrho^{(\theta)}} \frac{|u|^{2m}}{\varrho^2}\, dxdt \le \theta^{2m},$$

we have the Sobolev-Poincaré inequality

$$\iint_{Q_\varrho^{(\theta)}} \frac{\left|\boldsymbol{u}^m - (\boldsymbol{u}^m)_\varrho^{(\theta)}\right|^2}{\varrho^2}\, dx\, dt$$

$$\le \varepsilon \sup_{t\in\Lambda_\varrho^{(\theta)}} \int_{B_\varrho} \theta^{m-1} \frac{\left|\boldsymbol{u}^{\frac{m+1}{2}}(\cdot,t) - \left[(\boldsymbol{u}^m)_\varrho^{(\theta)}\right]^{\frac{m+1}{2m}}\right|^2}{\varrho^{\frac{m+1}{m}}}\, dx$$

$$+ \frac{c}{\varepsilon^{\frac{2}{n}}} \left[\iint_{Q_\varrho^{(\theta)}} \left|D\boldsymbol{u}^m\right|^{\frac{2n}{d}}\, dx\, dt\right]^{\frac{d}{n}} + c \iint_{Q_\varrho^{(\theta)}} |F|^2\, dx\, dt$$

for every $\varepsilon \in (0,1)$, where $(\boldsymbol{u}^m)_\varrho^{(\theta)} := \iint_{Q_\varrho^{(\theta)}} \boldsymbol{u}^m\, dxdt$.

Why there is no time derivative in the Poincaré inequality?

Because the differential equation yields the estimate

$$\left|\fint_{B_{\hat\varrho}} u(x,t_2)\,dx - \fint_{B_{\hat\varrho}} u(x,t_1)\,dx\right| \le \frac{c\,\varrho^{\frac{1}{m}}}{\theta^{m-1}} \iint_{Q_\varrho^{(\theta)}} \big(|D\boldsymbol{u}^m| + |F|\big)\,dxdt$$

for a "good radius" $\hat\varrho \in [\frac{\varrho}{2}, \varrho]$, which allows to replace the mean value

$$(\boldsymbol{u}^m)_\varrho^{(\theta)} \qquad \text{by} \qquad \left(\fint_{B_{\hat\varrho}} u(x,t)\,dx\right)^m$$

on the left-hand side of the Poincaré inequality, if $Q_\varrho^{(\theta)}$ is a *sub-intrinsic* cylinder.

Reverse Hölder inequality

Energy bounds and Poincaré inequality can be combined to a reverse Hölder inequality on appropriate cylinders. The key observation is that always one of the following two cases holds true:

The **degenerate case,** in which we have

$$\iint_{Q_{2\varrho}^{(\theta)}} \frac{|u|^{2m}}{(2\varrho)^2}\,dx\,dt \le \theta^{2m} \le K \iint_{Q_{\varrho}^{(\theta)}} \Big[\big|D\boldsymbol{u}^m\big|^2 + |F|^2\Big]\,dx\,dt \qquad \text{(D)}$$

for some constant $K \ge 1$, and the **non-degenerate case,** in which we have an intrinsic coupling of the form

$$\iint_{Q_{2\varrho}^{(\theta)}} \frac{|u|^{2m}}{(2\varrho)^2}\,dx\,dt \le \theta^{2m} \le \iint_{Q_{\varrho}^{(\theta)}} \frac{|u|^{2m}}{\varrho^2}\,dx\,dt. \qquad \text{(ND)}$$

Reverse Hölder inequality

Proposition (Bögelein, Duzaar, Korte, S.)

If u is a weak solution to the porous medium type system in Ω_T and $Q_{2\varrho}^{(\theta)} \Subset \Omega_T$ ($\varrho \leq 1$, $\theta > 0$) is a cylinder for which either (D) or (ND) holds true, then we have a reverse Hölder inequality of the form

$$\iint\!\!\!\!\!\!-_{Q_{\varrho}^{(\theta)}} \left|D\boldsymbol{u}^m\right|^2 dx\,dt$$

$$\leq c\left[\iint\!\!\!\!\!\!-_{Q_{2\varrho}^{(\theta)}} \left|D\boldsymbol{u}^m\right|^{\frac{2n}{d}} dx\,dt\right]^{\frac{d}{n}} + c\iint\!\!\!\!\!\!-_{Q_{2\varrho}^{(\theta)}} |F|^2\,dx\,dt.$$

Construction of sub-intrinsic cylinders – Step 1

The construction follows ideas by Schwarzacher (2014).

For $z_o \in Q_{2R} \subset \Omega_T$ and $\varrho \in (0, R]$, we define

$$\widetilde{\theta}_\varrho := \widetilde{\theta}_{z_o;\varrho} := \inf\left\{\theta \in [\lambda_o, \infty) : \iint_{Q_\varrho^{(\theta)}(z_o)} \frac{|u|^{2m}}{\varrho^2}\, dx\, dt \le \theta^{2m}\right\},$$

where $\lambda_o \gg 1$ is determined in dependence on u and F.

Then, $Q_\varrho^{\widetilde{\theta}_\varrho}$ is sub-intrinsic, and if $\widetilde{\theta}_\varrho > \lambda_o$, we even have

$$\iint_{Q_\varrho^{(\widetilde{\theta}_\varrho)}(z_o)} \frac{|u|^{2m}}{\varrho^2}\, dx\, dt = \widetilde{\theta}_\varrho^{2m}.$$

Problem:

There is not reason why $\varrho \mapsto \widetilde{\theta}_{z_o,\varrho}$ should be non-increasing, so that $Q_\varrho^{(\widetilde{\theta}_\varrho)} \not\subset Q_r^{(\widetilde{\theta}_r)}$ might hold for $\varrho < r$.

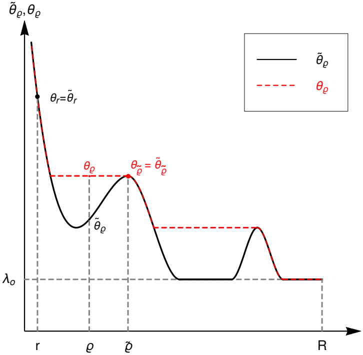
Construction of sub-intrinsic cylinders – Step 2

The solution is a
Rising-sun construction:

We replace $\widetilde{\theta}_\varrho$ by

$$\theta_\varrho := \theta_{z_o;\varrho} := \max_{r\in[\varrho,R]} \widetilde{\theta}_{z_o;r}\,.$$

Then the cylinders $Q_\varrho^{(\theta_\varrho)}(z_o)$ are still **sub-intrinsic** since



$$\iint_{Q_\varrho^{(\theta_\varrho)}} \frac{|u|^{2m}}{\varrho^2}\,dx\,dt \le \left(\frac{\theta_\varrho}{\widetilde{\theta}_\varrho}\right)^{m-1} \iint_{Q_\varrho^{(\widetilde{\theta}_\varrho)}} \frac{|u|^{2m}}{\varrho^2}\,dx\,dt$$
$$\le \left(\frac{\theta_\varrho}{\widetilde{\theta}_\varrho}\right)^{m-1} \widetilde{\theta}_\varrho^{2m} = \theta_\varrho^{m-1}\widetilde{\theta}_\varrho^{m+1} \le \theta_\varrho^{2m}.$$

Properties of θ_ϱ

- Since $\varrho \mapsto \theta_\varrho$ is non-increasing, the cylinders for a fixed center z_o are nested in the sense

$$Q_\varrho^{(\theta_\varrho)}(z_o) \subset Q_r^{(\theta_r)}(z_o) \qquad \text{provided } \varrho < r.$$

- For $\varrho < r$, we have $\theta_\varrho \le \left(\frac{r}{\varrho}\right)^{\frac{d+2}{m+1}} \theta_r$.
- The map $\varrho \mapsto \theta_\varrho$ is continuous.
- The family of cylinders $\big(Q_\varrho^{(\theta_{z_o;\varrho})}(z_o)\big)_{\varrho, z_o}$ has a Vitali type covering property.

Stopping time argument with sub-intrinsic cylinders

We implement a stopping time argument as for the p-Laplacean, but now with the sub-intrinsic cylinders $Q_\varrho^{(\theta_\varrho)}(z_o)$:

For $\lambda \gg \lambda_o$ and $z_o \in E(\lambda) := \{z \in \Omega_T \,:\, |D\boldsymbol{u}^m|(z) > \lambda^m\}$, we consider the mean value

$$\varrho \mapsto \iint\!\!\!\!\!\!\!-_{Q_\varrho^{(\theta_\varrho)}(z_0)} \left(|D\boldsymbol{u}^m|^2 + |F|^2\right) dx\, dt,$$

which depends continuously on ϱ, is $> \lambda^{2m}$ for ϱ small and $< \lambda^{2m}$ for ϱ large. Hence, we can choose a maximal radius ϱ_{z_o} with

$$\iint\!\!\!\!\!\!\!-_{Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} \left(|D\boldsymbol{u}^m|^2 + |F|^2\right) dx\, dt = \lambda^{2m}.$$

Conclusion of the construction

The result of the construction is a family of parabolic cylinders $Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$, $z_o \in \Omega_T$, with

$$\iint_{Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)} \left(|D\boldsymbol{u}^m|^2 + |F|^2\right) dx\, dt = \lambda^{2m}.$$

This means that we have constructed a family of *sub-intrinsic* cylinders

- whose geometry is determined by the porous medium equation (i.e. by the size of $|u|$),
- and for which the mean value is coupled to λ^{2m} (which is related to $|D\boldsymbol{u}^m|^2$).

This settles the problem to combine the two different parameters θ and λ.

- On the cylinder $Q_{\varrho_{z_o}}^{(\theta_{\varrho_{z_o}})}(z_o)$, we obtain a reverse Hölder inequality (by distinguishing between the degenerate and the non-degenerate case).
- Using the Vitali type covering property of the cylinders, we get a countable covering of the super-level sets $E(\lambda)$ by such cylinders.
- Then a standard Fubini type argument yields a higher integrability estimate of the form

$$\begin{aligned}
&\iint_{Q_R} \big|D\boldsymbol{u}^m\big|^{2+\varepsilon}\,dx\,dt \\
&\qquad \le c\left[1+\iint_{Q_{2R}}\left[\frac{|u|^{2m}}{R^2}+|F|^2\right]dx\,dt\right]^{\frac{\varepsilon m}{m+1}}\iint_{Q_{2R}}\big|D\boldsymbol{u}^m\big|^2\,dx\,dt \\
&\qquad\quad + c\iint_{Q_{2R}}|F|^{2+\varepsilon}\,dx\,dt,
\end{aligned}$$

provided $Q_{2R} := Q_{2R}^{(1)} \Subset \Omega_T$.

Further applications: doubly non-linear systems

The techniques are robust enough to be applied to various other settings, e.g. to *doubly non-linear systems*, whose model case is

$$\partial_t \boldsymbol{v}^m - \operatorname{div}\big(|Dv|^{p-2}Dv\big) = 0.$$

- For $m = 1$, this is the parabolic p-Laplace system;
- for $p = 2$, this is the porous medium system (with the transformation $v := \boldsymbol{u}^m$ and $m \rightsquigarrow \frac{1}{m}$.)

This is work in progress. The higher integrability

$$|Dv| \in L^{p+\varepsilon}_{\mathrm{loc}}(\Omega_T)$$

has already been established in the doubly-degenerate case $0 < m < 1$ and $p > 2$.

In the doubly-degenerate case $0 < m < 1$ and $p > 2$, the modulus of ellipticity of the system

$$\partial_t \boldsymbol{v}^m - \mathrm{div}\big(|Dv|^{p-2}Dv\big) = 0.$$

degenerates both if $|u|$ becomes small and if $|Du|$ becomes small. Therefore, both quantities have to be taken into account in the geometry of the cylinders, which now take the form

$$Q_\varrho^{(\lambda,\theta)} := B_\varrho \times \big(-\lambda^{2-p}\theta^{m-1}\varrho^{1+m}, \lambda^{2-p}\theta^{m-1}\varrho^{1+m}\big),$$

where (heuristically)

$$\theta \approx \frac{|v|}{\varrho} \qquad \text{and} \qquad \lambda \approx |Dv|.$$

The strategy for the construction of suitable cylinders is now:

1. For a fixed λ and $z_o \in \Omega_T$, construct the parameters $\theta_\varrho^{(\lambda)}$ by the rising-sun construction. This yields a family of nested sub-intrinsic cylinders
$$Q_\varrho^{(\lambda,\theta_\varrho^{(\lambda)})}(z_o).$$
2. For $\lambda \geq \lambda_o \gg 1$ and z_o with $|Dv(z_o)| > \lambda$, construct radii ϱ_{z_o} by a stopping time argument.

The result are cylinders $Q_{\varrho_{z_o}}^{(\lambda,\theta_{\varrho_{z_o}}^{(\lambda)})}(z_o)$ for which

$$\iint_{Q_{\varrho_{z_o}}^{(\lambda,\theta_{\varrho_{z_o}}^{(\lambda)})}(z_o)} \big(|Dv|^p + |F|^p\big)\, dx\, dt = \lambda^p.$$

Further applications: non-divergence inhomogeneities

Now, we consider (possibly signed) solutions to equations of the form

$$\partial_t u - \operatorname{div} A(x,t,u,D\boldsymbol{u}^m) = f \qquad \text{in } \Omega_T.$$

For this problem, we consider the initial regularity condition $|D\boldsymbol{u}^{\frac{m+1}{2}}| \in L^2(\Omega_T)$ for the solution, because this allows to consider right-hand sides with lower integrability exponent.

What is the optimal regularity for f? The conditions $u \in C^0([0,T];L^2(\Omega))$ with $\boldsymbol{u}^{\frac{m+1}{2}} \in L^2(0,T;W^{1,2}(\Omega))$ imply

$$u \in L^{m_\diamond}(\Omega_T) \qquad \text{with } m_\diamond := m+1+\frac{4}{n}.$$

Hence, $f \in L^{m_\diamond'}(\Omega_T)$ with $m_\diamond' = \frac{n(m+1)+4}{nm+4}$ is the minimal regularity to guarantee

$$\int_{\Omega_T} |u|\,|f|\,dx\,dt < \infty.$$

Theorem (Bögelein, Duzaar, S., Singer, 2018)

Let $u : \Omega_T \to \mathbb{R}$ be a weak solution of the porous medium type equation

$$\partial_t u - \operatorname{div} A(x, t, u, D\boldsymbol{u}^m) = f, \qquad \text{in } \Omega_T$$

with $f \in L^\sigma(\Omega_T)$ for $\sigma > \frac{n(m+1)+4}{nm+4} = m'_\diamond$. Then we have

$$|D\boldsymbol{u}^{\frac{m+1}{2}}| \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega_T) \qquad \text{for some } \varepsilon > 0.$$

Thank you for your attention!