The complex Riccati equation

Matias Dahl

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In the below, all matrices are assumed to be complex $n \times n$ matrices unless otherwise mentioned. By a positive definite matrix we mean a real symmetric matrix A such that $\eta^T \cdot A \cdot \eta > 0$ for all $\eta \in \mathbb{R}^n \backslash \{0\}$, or equivalently, if $\eta^T \cdot A \cdot \eta > 0$ for all $\eta \in \mathbb{C}^n \backslash \{0\}$. If A is a matrix we denote the transpose, conjugate transpose, and imaginary part of A by A^T, A^*, and $\operatorname{Im} A$, respectively. By $\langle \, , \, \rangle$ we denote the usual inner product in $\mathbb{C}^n$; for $\eta = (\eta_1, \ldots, \eta_n)$, $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$,

$$\langle \eta, \zeta \rangle = \sum_{k=1}^n \eta_k \overline{\zeta_k}.$$

Suppose $I \ni 0$ is an open interval, and suppose B, C, D are real matrices depending continuously on $t \in I$, suppose C, D are symmetric, and all matrices are bounded on I. Under these assumptions, we shall study the *Riccati equation*

$$\frac{dH}{dt} + BH + HB^T + HCH + D = 0. \tag{1}$$

Following [1], we shall prove the following result:

Proposition 0.1. *Suppose H_0 is a symmetric $n \times n$ matrix such that $\operatorname{Im} H_0$ is positive definite. Then equation 1 has a unique solution H on I such that*

1. *$H(0) = H_0$,*
2. *H is symmetric and $\operatorname{Im} H$ is positive definite for all t.*

To prove this we first study the linear differential equation,

$$\frac{d}{dt}\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} D & B \\ B^T & C \end{pmatrix}\begin{pmatrix} Y \\ Z \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} Y \\ Z \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix}. \tag{3}$$

Here Y, Z are unknown matrices depending on $t \in I$, and the initial value (Y_0, Z_0) is known. In [2] it is proven that equation 2 has a global solution on I and this solution is uniquely determined by the initial condition (Y_0, Z_0).

Lemma 0.2. *Suppose Y_0, Z_0 are matrices such that Y_0 is invertible, $Z_0 Y_0^{-1}$ is symmetric, and $\operatorname{Im}(Z_0 Y_0^{-1})$ is positive definite. Further, suppose (Y, Z) is the solution to equations 2, 3 with initial value (Y_0, Z_0). Then matrices*

$$Z^T Y - Y^T Z, \qquad Z^* Y - Y^* Z$$

are independent of $t \in I$, and Y is invertible for all t.

Proof. Let $\tau \in \{T, *\}$. Since (Y, Z) satisfies equation 2, and B, C, D are real matrices, we have

$$\frac{d}{dt}\left(\begin{pmatrix} Y \\ Z \end{pmatrix}^{\tau} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} \right) = 0,$$

and the first two claims follows. Let us next show that Y is invertible for all $t \in I$. For a contradiction, suppose $Y(s)\eta = 0$ for some $s \in I$, $\eta \in \mathbb{C}^n$. Then

$$\begin{aligned} \langle (Z_0^* Y_0 - Y_0^* Z_0)\eta, \eta \rangle &= \langle (Z^* Y - Y^* Z)(s)\eta, \eta \rangle \\ &= 0. \end{aligned}$$

For a complex number $\alpha \in \mathbb{C}$, we have $\operatorname{Im} \alpha = \frac{i}{2}(\overline{\alpha} - \alpha)$, so for any $\zeta \in \mathbb{C}^n$,

$$\langle \operatorname{Im}(Z_0 Y_0^{-1})\zeta, \zeta \rangle = \frac{i}{2} \langle (Z_0^* Y_0 - Y_0^* Z_0) Y_0^{-1}\zeta, Y_0^{-1}\zeta \rangle.$$

Putting $\zeta = Y_0 \eta$ gives a contradiction with the assumption on $\operatorname{Im}(Z_0 Y_0^{-1})$. □

The next lemma shows how equations 1 and 2 are related.

Lemma 0.3.

1. *Suppose Y_0, Z_0, Y, Z are as in Lemma 0.2. Then*

   $$H = Z Y^{-1}$$

   is a solution to equation 1 for all $t \in I$ with initial value $H(0) = Z_0 Y_0^{-1}$.

2. *Let H be a solution to equation 1 with initial value H_0. Then there exist matrices Y_0, Z_0 such that Y_0 is invertible,*

   $$H_0 = Z_0 Y_0^{-1},$$

   and

   $$H = Z Y^{-1},$$

   when (Y, Z) is the solution to equations 2, 3, with initial value (Y_0, Z_0).

Proof. For any matrix invertible and differentiable for all $t \in I$, we have

$$\frac{dA^{-1}}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}.$$

Thus, if (Y, Z) is a solution to equation 2, then

$$\frac{d}{dt}(ZY^{-1}) = -BH - HB^T - HCH - D.$$

This formula implies the first claim. In the second claim, $Y_0 = I$, $Z_0 = H_0$ is a suitable choice of Y_0, Z_0. The representation $H = ZY^{-1}$ follows since $(H - ZY^{-1})(0) = 0$, and $\frac{d}{dt}(H - ZY^{-1}) = 0$. □

Proof of Proposition 0.1. Since $H_0 = Z_0Y_0^{-1}$ for the choice $Y_0 = I$, $Z_0 = H_0$, Lemma 0.3.1 implies that equation 1 has a global solution. For uniqueness, suppose $H, \tilde{H}$ are two solutions to equation 1 satisfying

$$H(0) = \tilde{H}(0) = H_0.$$

Then Lemma 0.3.2 implies that there are matrices $Y_0, \tilde{Y}_0, Z_0, \tilde{Z}_0$ such that

$$H_0 = Z_0Y_0^{-1} = \tilde{Z}_0\tilde{Y}_0^{-1}, \tag{4}$$

and

$$H = ZY^{-1}, \quad \tilde{H} = \tilde{Z}\tilde{Y}^{-1},$$

where (Y, Z) and $(\tilde{Y}, \tilde{Z})$ are the solutions to equation 2 with initial values (Y_0, Z_0), $(\tilde{Y}_0, \tilde{Z}_0)$, respectively. We know that solutions to equation 2 are uniquely determined by the initial value. Thus

$$\tilde{Y} = Y(Y_0^{-1}\tilde{Y}_0), \quad \tilde{Z} = Z(Y_0^{-1}\tilde{Y}_0). \tag{5}$$

Indeed, as $(Y_0^{-1}\tilde{Y}_0)$ is a constant matrix, the right hand sides in equation 5 satisfy equation 2 and the same initial condition as $(\tilde{Y}, \tilde{Z})$. By equation 4,

$$\tilde{H} = \tilde{Z}\tilde{Y}^{-1} = ZY^{-1} = H,$$

and the solution to equation 1 is unique. Suppose H is a solution to equation 1. Then H^T is also a solution to equation 1, and since H and H^T share the same initial value, H is symmetric. It remains to prove that $\operatorname{Im} H$ is positive definite. Since $\operatorname{Im} H_0$ is symmetric, it follows that $\operatorname{Im} H_0 = \frac{i}{2}(H_0^* - H_0)$, so

$$\operatorname{Im} H_0 = \frac{i}{2}(Y_0^{-1})^*(Z_0^*Y_0 - Y_0^*Z_0)Y_0^{-1}.$$

and since $\operatorname{Im} H$ is symmetric,

$$\begin{aligned} \operatorname{Im} H &= \frac{i}{2}(Y^{-1})^*(Z^*Y - Y^*Z)Y^{-1} \\ &= (Y_0Y^{-1})^* \operatorname{Im}(H_0)(Y_0Y^{-1}) \end{aligned}$$

by Lemma 0.2. □

References

[1] A. Kachalov, Y. Kurylev, M. Lassas, *Inverse Boundary Spectral Problems*, Chapman & Hall/CRC, 2001.

[2] M. E. Taylor, *Partial Differential Equations: Basic Theory*, Springer, 1996.