Experiments with moduli of quadrilaterals

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Abstract. Basic facts and definitions of conformal moduli of rings and quadrilaterals are recalled. Some computational methods are reviewed. For the case of quadrilaterals with polygonal sides, some recent results are given. Some numerical experiments are presented. This paper is based on [BSV] and [RV].

1. Introduction

We give a brief introduction to the conformal moduli of quadrilaterals and rings. For a comprehensive survey of this topic see [Küh].

The capacity of condensers has been studied because of its importance in physics and its close relation with the potential theory and the theory of conformal and quasiconformal mappings. The analytic computation of capacity is possible only for very few types of condensers and for this reason several methods have been developed for the numerical computation of capacity.

Let E and F be two disjoint compact sets in the extended complex plane $\overline{\mathbb{C}}$. We assume that each of E and F is the union of a finite number of nondegenerate disjoint continua, and that the open set $R = \overline{\mathbb{C}} \setminus (E \cup F)$ is connected. Without loss of generality, we also assume that $\infty \notin E$. The domain R is a *condenser*. The complementary compact sets E and F are the *plates* of the condenser. The *capacity* of R is defined by

$$\operatorname{cap} R = \inf_u \int_R |\nabla u|^2 \, dm, \tag{1.1}$$

where the infimum is taken over all nonnegative, piecewise differentiable functions u with compact support in $R \cup E$ such that $u = 1$ on E. It is well-known that under the assumptions we made above, R is regular for the Dirichlet problem and the harmonic function on R with boundary values 1 on E and 0 on F is the unique function that minimizes the integral in (1.1). This function is called the *potential function* of the condenser.

Capacity is a conformal invariant: Suppose that f maps R conformally onto R'. Let E and F correspond to E' and F' respectively (in the sense of the boundary correspondence under conformal mapping). Then $\operatorname{cap} R = \operatorname{cap} R'$. This property can be used for the analytic computation of capacity provided that the capacity of some 'canonical' condensers is known and the corresponding conformal mappings can be constructed. Unfortunately such an analytic computation can be made only for very few doubly-connected condensers; see [IT].

If both E and F are connected (and hence R is doubly-connected), R is called a *ring domain.* A ring domain R can be mapped conformally onto the annulus $\{z : 1 < |z| < e^M\}$, where $M = \operatorname{mod} R$ is the *conformal modulus* of the ring domain R, defined by $\operatorname{mod} R = 2\pi/\operatorname{cap} R$. See also [Ahl], [Hen], [Küh].

A Jordan domain D in $\mathbb{C}$ with marked points $z_1, z_2, z_3, z_4 \in \partial D$ is a quadrilateral and denoted by $(D; z_1, z_2, z_3, z_4)$. We use the canonical map onto a rectangle $(D'; 0, 1, 1+ih, ih)$ to define the modulus h of a quadrilateral $(D; z_1, z_2, z_3, z_4)$. The modulus of $(D; z_2, z_3, z_4, z_1)$ is $1/h$. We mainly study the situation where the boundary of D consists of the polygonal line segments through z_1, z_2, z_3, z_4 (always positively oriented). In this case, the modulus is denoted by $\mathrm{QM}(D; z_1, z_2, z_3, z_4)$. If the boundary of D consists of straight lines connecting the given boundary points, we omit the domain D and denote the quadrilateral and the corresponding modulus simply by (z_1, z_2, z_3, z_4) and $\mathrm{QM}(z_1, z_2, z_3, z_4)$.

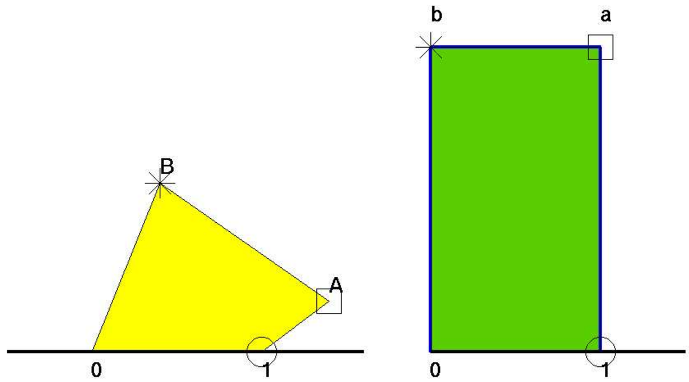

FIGURE 1: The height of the canonical rectangle is $\mathrm{QM}(A, B, 0, 1)$.

The following problem is known as the *Dirichlet-Neumann problem.* Let D be a region in the complex plane whose boundary ∂D consists of a finite number of regular Jordan curves, so that at every point of the boundary a normal is defined. Let ψ to be a real-valued continuous function defined on ∂D. Let $\partial D = A \cup B$ where A, B both are unions of Jordan arcs. Find a function u satisfying the following conditions:

(1) u is continuous and differentiable in $\overline{D}$.
(2) $u(t) = \psi(t), \qquad t \in A$
(3) If $\partial/\partial n$ denotes differentiation in the direction of the exterior normal, then

$$\frac{\partial}{\partial n} u(t) = \psi(t), \qquad t \in B.$$

One can express the modulus of a quadrilateral $(D; z_1, z_2, z_3, z_4)$ in terms of the solution of the Dirichlet-Neumann problem as follows. Let $\gamma_j, j = 1, 2, 3, 4$ be the arcs of ∂D between (z_1, z_2), (z_2, z_3), (z_3, z_4), (z_4, z_1), respectively. If u is the (unique) harmonic solution of the Dirichlet-Neumann problem with boundary values equal to 0 on γ_2, equal to 1 on γ_4 and with $\partial u/\partial n = 0$ on $\gamma_1 \cup \gamma_3$, then by [Ahl, p.65/Thm 4.5]:

$$\mathrm{QM}(D; z_1, z_2, z_3, z_4) = \int_D |\nabla u|^2 \, dm. \tag{1.2}$$

In conclusion, the computation of the modulus of a ring or quadrilateral can be reduced to solving the Dirichlet problem (1.1) or the Dirichlet-Neumann problem, respectively. The connection between ring and quadrilateral moduli is given in [Küh, p.102] or [LV, p.36].

2. Review of some numerical methods

Both for the ring and the quadrilateral case we may consider the following methods:

(a) Approximate the canonical conformal map.
(b) Approximate the solution of the Dirichlet (or Dirichlet-Neumann) problem.

The recent survey of Wegmann [Weg] provides an extensive review of the various techniques of the approximation of conformal mappings. See also Driscoll and Trefethen [DrTr], Papamichael [Pap]. We now mention some of the known methods, following closely [BSV].

The paper [Gai] of D. Gaier includes a review of the various methods applicable to the computation of the capacity of planar ring domains.

The finite element method was first applied to the computation of capacity by G. Opfer [Opf]. Several numerical experiments are reported by J. Weisel [Wei2]. Another numerical method is based on the Gauss-Thompson principle which implies a formula for the capacity involving Green's function. Numerical computations are given in [Wei1]. N. Papamichael and his collaborators [PKo], [PS], [PWa] have developed an orthonormalization technique for the approximation of the conformal mapping of doubly-connected domains. This technique gives, in particular, approximations of capacity. Many numerical computations are presented in the above papers.

The capacity of a polygonal ring domain can be also computed by the Schwarz-Christoffel transformation which provides a semi-explicit formula for the conformal mapping of the domain onto an annulus (see [Hen]). For simply-connected domains this methods has been developed by T. Driscoll, L.N. Trefethen and their coauthors (see [DrTr], [DrVa]). It seems that for doubly-connected polygonal domains the only related works are those of H. Daeppen [Dae] and C. Hu [Hu]. Hu's method has been tested successfully in several computations, (see [Hu], [BV]). It is partially based on the wise choice of certain points on the complementary sets E, F of the ring domain.

Another numerical-analytic method that can be used for the computation of capacity is the multipole method. The potential function is written as a linear combination of explicit basic functions (multipoles) with unknown coefficients. The coefficients are then computed numerically. The multipoles constitute a complete, minimal system in a certain Hardy-type space of functions. Their construction is based on the theory of conformal mapping. This method has been developed by V.I. Vlasov (see [Vla] and references therein) as a general method for numerical solution of a wide class of boundary value problems. He has applied this approach to find the potential function of condensers.

3. Web-based simulations

The solutions of the Dirichlet and the Dirichlet-Neumann problems can be approximated by the method of finite elements, see [Hen, pp. 305–314], [Pap]. Hence, this method can also be used to approximate the modulus of quadrilaterals and rings.

The Dirichlet-Neumann problem can be numerically solved with AFEM (Adaptive FEM) numerical PDE analysis package by Klas Samuelsson. This software applies, e.g., to multiply connected polygonal domains. In particular, we may use it to compute the modulus (capacity) of a bounded ring whose boundary components are broken lines. Examples and applications for this software are given in [BSV]. In [HVV] a theoretical formula for computing $\mathrm{QM}(A,B,0,1)$ was given with its implementation with Mathematica. This lead to a more systematic study of the modulus of quadrilateral in [DuVu]. In the course of the work on [DuVu] several conjectures were formulated and this lead us to look for an improved version of the algorithm in [HVV] for the computation of $\mathrm{QM}(A,B,0,1)$. It seems that the AFEM software of Samuelsson is very efficient for this purpose.

Our goal is to use the AFEM software of Samuelsson for computations involving moduli of polygonal rings and quadrilaterals, and to write a user-interface providing access to AFEM via a web browser (e.g. Mozilla). The advantages in this approach are: (1) no programming needed to use AFEM and (2) mobile computing: available for everyone. Currently the pilot tests work locally at the local university network.

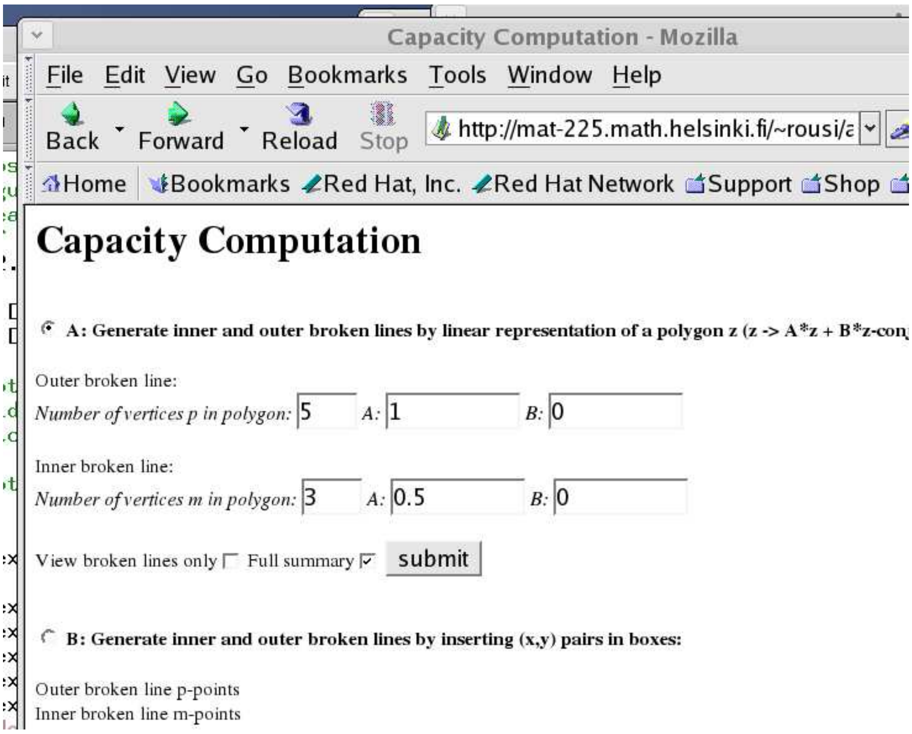

Figure 2: Entering two regular polygons for computing the capacity of the corresponding ring domain by using a web browser.

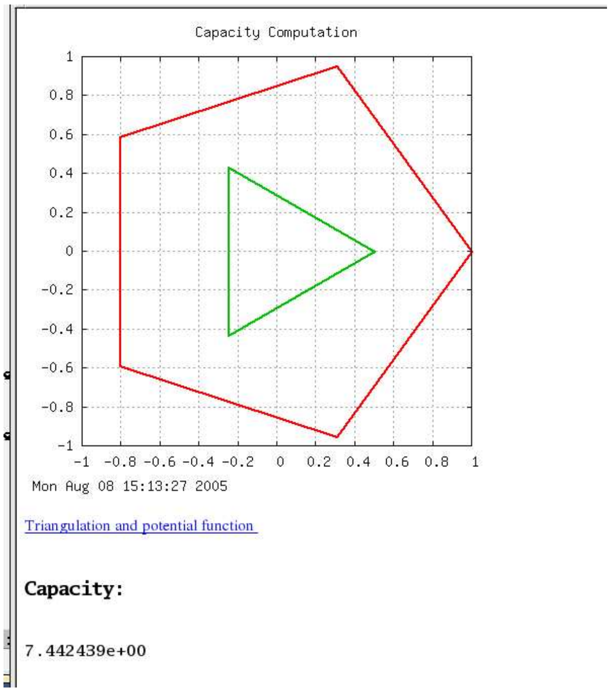

FIGURE 3: The output of the program is also displayed by using the web-based interface.

4. EXPERIMENTS

We give some examples of problems which can be studied by using the AFEM software.

4.1. Example. Let $a, b \in C$ with $\operatorname{Im} a > 0, \operatorname{Im} b > 0$ and assume that $(a, b, 0, 1)$ determines the vertices of a quadrilateral and $\arg b \in (\pi/2, \pi)$, $\arg(a-1) \in (0, \pi/2)$. Is it true that

(4.2) $$\mathrm{QM}(a, b, 0, 1) \le \mathrm{QM}(1 + i|a-1|, i|b|, 0, 1)?$$

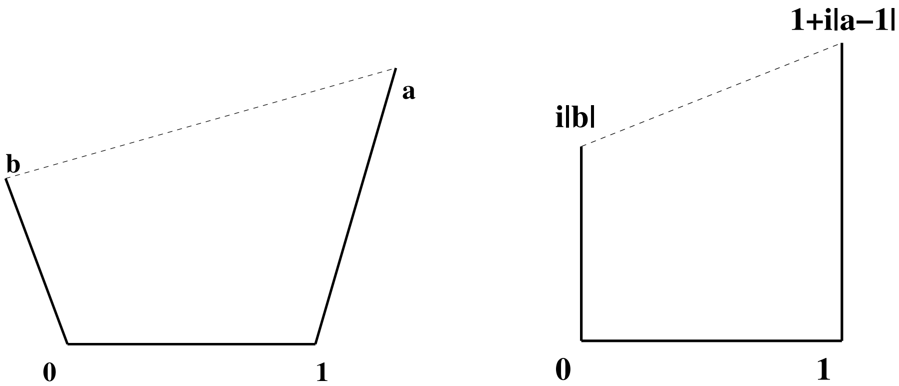

FIGURE 4: The quadrilaterals $(a, b, 0, 1)$ and $(1 + |a-1|i, |b|i, 0, 1)$ for the left and right sides for (4.2), respectively.

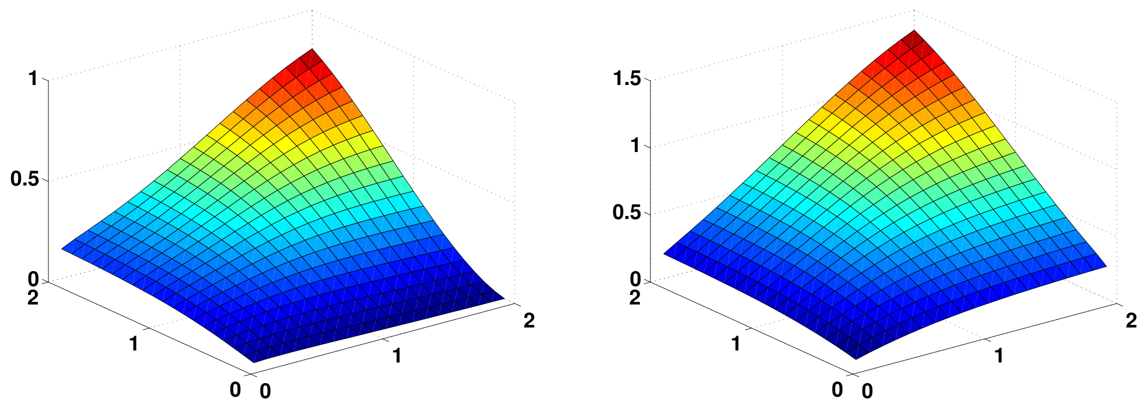

FIGURE 5: The function $f(x,y) = \mathrm{QM}(1+xi, yi, 0, 1) - \mathrm{QM}(1+x\exp(i\alpha), y\exp(i\beta), 0, 1)$ for $x, y \in (0,2)$. Here $\beta = 3\pi/4$ and α is $\pi/2$ (left) and $\pi/8$ (right).

4.3. Remark. The expression on the right hand side of (4.2) has an analytic expression if $|a-1| = h = |b|+1$. Bowman [Bow, pp. 103-104] gives a formula for the conformal modulus of the quadrilateral with vertices $1+hi$, $(h-1)i$, 0, and 1 when $h > 1$ as $M(h) \equiv \mathcal{K}(r)/\mathcal{K}(r')$ where

$$r = \left(\frac{t_1 - t_2}{t_1 + t_2}\right)^2, \quad t_1 = \mu^{-1}\left(\frac{\pi}{2c}\right), \quad t_2 = \mu^{-1}\left(\frac{\pi c}{2}\right), \quad c = 2h-1\,.$$

Here for $0 < r < 1$

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}},$$

and $r' = \sqrt{1-r^2}$.

Therefore, the quadrilateral can be conformally mapped onto the rectangle $1 + iM(h)$, $iM(h)$, 0, 1, with the vertices corresponding to each other. It is clear that $h - 1 \le M(h) \le h$. The formula

$$M(h) = h + c + O(e^{-\pi h}), c = -1/2 - \log 2/\pi\,,$$

is given in [PS]. As fas as we know there is neither an explicit nor asymptotic formula for the case when the angle $\pi/4$ of the trapezoid is equal to $\alpha \in (0, \pi/2)$.

4.4. Example. (Duplication formula) Let $\phi \in (0,\pi)$, $h, k > 0$, $A = 1 + h\exp(i\phi)$, $B = k\exp(i\phi)$. We study when the following inequality holds:

$$\mathrm{QM}(A,B,0,1) + \mathrm{QM}(\overline{(1-B)}, \overline{(1-A)}, 0, 1) \le \mathrm{QM}(A,B,1-A,1-B). \tag{4.5}$$

Here equality holds if $A = 1+ih$, $B = ih$ and $h > 0$. In this special case the result may be regarded as a duplication formula.

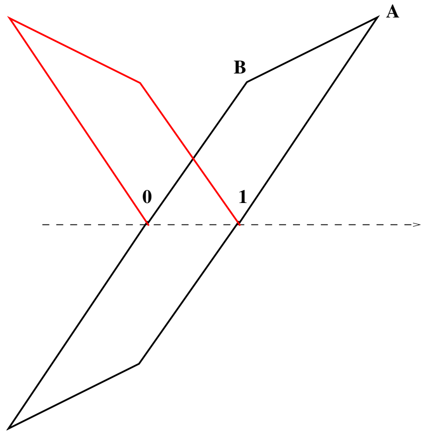

FIGURE 6: The quadrilaterals in (4.5).

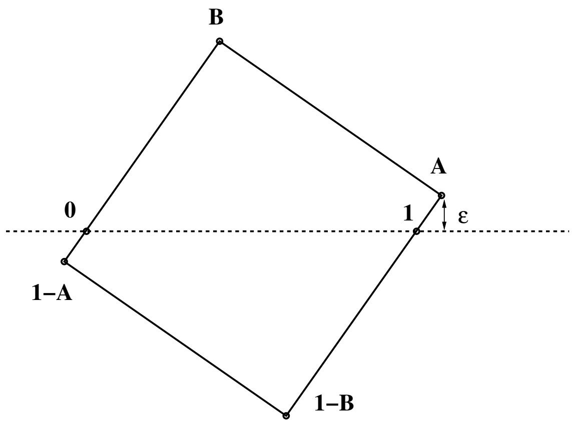

FIGURE 7: If $B = 1/2 + i(A - 1/2)$, then $A, B, 1-A, 1-B$ are the vertices of a square and hence $\mathrm{QM}(A, B, 1-A, 1-B) = 1$. Letting $|A-1| \to 0$ we see that the left side of (4.5) tends to 0 whereas the right side $= 1$.

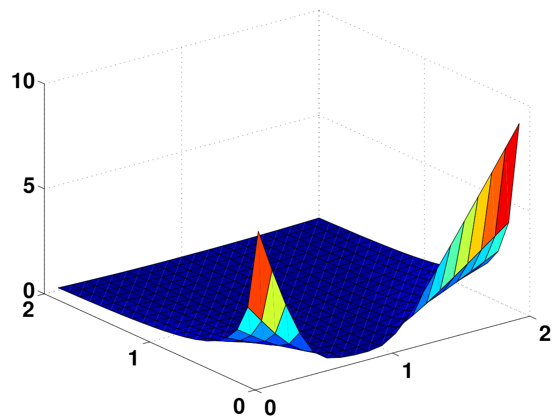

FIGURE 8: The function

$$g(x, y) = \mathrm{QM}(A, B, 1-A, 1-B) - \mathrm{QM}(A, B, 0, 1) - \mathrm{QM}(\overline{(1-B)}, \overline{(1-A)}, 0, 1)$$

where $A = x + iy$ and $B = \exp(i \arg(A-1))$ for $x, y \in (0, 2)$.

4.6. Remark. Let $h, k > 1$ and consider next the rectangle with the vertices $A = 1 + i(h + k - 1), B = i(h + k - 1), 0, 1$. If we split this rectangle in two trapezoids with the segment joining $i(h - 1)$ and $1 + ih$ and apply (4.5), then we get

$$\mathrm{QM}(A, B, 0, 1) \geq \mathrm{QM}(A, B, i(h - 1), 1 + ih) + \mathrm{QM}(0, 1, 1 + ih, i(h - 1)).$$

If we use the notation from 4.3 and use the formula $\mathrm{QM}(A, B, 0, 1) = h + k - 1$ then we can express this as

$$h + k - 1 \geq M(h) + M(k) \geq h + k - 2.$$

4.7. Example. (Open problem [DuVu]) Fix $r, s > 0, \alpha \in (0, \pi/2), \beta \in (\pi/2, \pi)$. Determine $t > 0$ by the condition that the quadrilaterals $Q_1 = (1 + 2re^{i\alpha}, 2se^{i\beta}, 0, 1)$ and $Q_2 = (t + ir, is, -is, t - ir)$ have equal areas. Is it true that

$$QM(1 + 2re^{i\alpha}, 2se^{i\beta}, 0, 1) \leq QM(t + ir, is, -is, t - ir)\,? \tag{4.8}$$

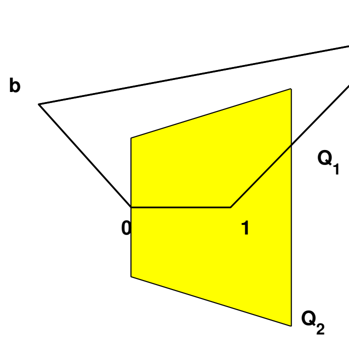

FIGURE 9: Quadrilaterals Q_1 and Q_2 of Example 4.7.

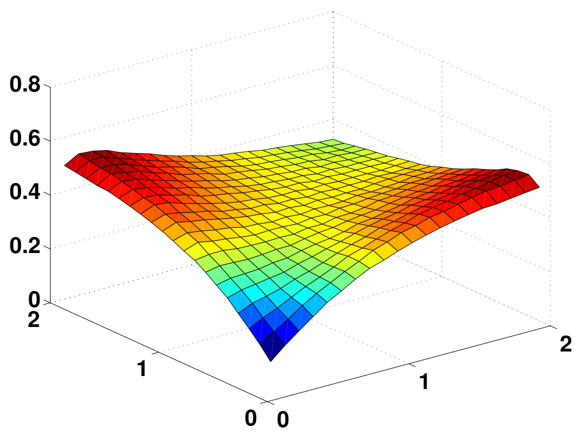

FIGURE 10: Function

$$h(r, s) = QM(t + ir, is, -is, t - ir) - QM(1 + 2re^{i\alpha}, 2se^{i\beta}, 0, 1)$$

for $\alpha = \pi/4$, $\beta = 3\pi/4$.

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