ANGULAR AND APPROXIMATE LIMITS OF QUASIREGULAR MAPPINGS

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Abstract. Concepts of angular and approximate limit are introduced. We give a brief introduction to these and related concepts in the context of quasiregular mappings of $\mathbb{R}^n$. The main goal in the study of this topic is to establish criteria for a quasiregular mapping to have angular and approximate limits at a given boundary point. We prove a version of the Schwarz lemma for quasiregular mappings by using results of P. Järvi, S. Rickman and M. Vuorinen. In the main result, we give a criterion for a quasiregular mapping to have a limit through a set of large upper measure density. In the last section, we study the angular limits of quasiregular mappings by applying Vuorinen's results involving the Harnack inequality.

1. Introduction

A classical result by E. Lindelöf states that a conformal mapping of $\mathbf{B}^2$ having an asymptotic value α at a boundary point b also has an angular limit α at b. A similar result for quasiconformal mappings in $\mathbb{R}^3$ was proved by F. Gehring in [2]. Another classical result, by P. Koebe, shows that if a bounded analytic function f tends to zero along a sequence of arcs in the unit disk which approach a subarc in the boundary, then f must be identically zero. Refinements of Koebe's results were given by D. Rung in [7].

The theory of quasiregular mappings gives a natural generalization for the geometric aspects of the theory of analytic functions in the complex plane. On the other hand, it can also be understood as a generalization of the theory of quasiconformal mappings. In light of these results, it is natural to ask what results of this type hold for quasiregular mappings. These topics were studied by M. Vuorinen in [8], [10] and [11, Chapter 15]. Besides angular limits, other concepts of limit which are useful in this setting are known. In [9] Vuorinen studied relations between these concepts in the more general setting of Harnack functions.

The results stated above lead us to the following three questions:

(1) Under which conditions do the limit concepts studied in [9] imply each other?
(2) Can the results concerning angular limits proved in [8] and [10] be improved with more careful analysis of the mapping involved? In particular, what kind of role does the local topological index of the mapping play in the boundary behavior?
(3) What results do we have for quasiregular mappings in the case of the other limit concepts, such as approximate limits?

In this article we present the notions of angular and approximate limits. Then we give a brief introduction to the methods used to study questions (2) and (3). In the main result, Theorem 3.5, we give a condition for a quasiregular mapping to have a limit through a set of large upper measure density, in terms of the local

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topological index. More results of this type are given in [5, pp. 48–53]. In Section 4 we give an example of how the techniques from [9] can be applied in the context of quasiregular mappings. A thorough investigation of the question (1) can be found in [5, pp. 36–45].

2. Notation and preliminaries

We shall follow standard notation and terminology adopted from [6] and [11]. For $x \in \mathbb{R}^n$, $n \geq 2$, and $r > 0$ let $\mathbf{B}^n(x,r) = \{z \in \mathbb{R}^n : |z-x| < r\}$, $S^{n-1}(x,r) = \partial\mathbf{B}^n(x,r)$, $\mathbf{B}^n(r) = \mathbf{B}^n(0,r)$, $S^{n-1}(r) = \partial\mathbf{B}^n(r)$, $\mathbf{B}^n = \mathbf{B}^n(1)$, $\mathbf{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$, $\mathbf{B}^n_+ = \mathbf{B}^n \cap \mathbf{H}^n$, and $S^{n-1} = \partial\mathbf{B}^n$. The standard coordinate unit vectors are denoted by $e_1, \ldots, e_n$. Lebesgue measure on $\mathbb{R}^n$ is denoted by m.

The hyperbolic metrics in the upper half space $\mathbf{H}^n$ or the unit ball $\mathbf{B}^n$ (see [11, pp. 21, 23.]) are denoted by $\rho(x,y)$. For the half space $\mathbf{H}^n$ we have the following formula for the hyperbolic metric:

$$\cosh\rho(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n}, \qquad x, y \in \mathbf{H}^n. \tag{2.1}$$

For a in $\mathbf{H}^n$ or in $\mathbf{B}^n$ and $M > 0$ the hyperbolic ball $\{x : \rho(a,x) < M\}$ is denoted by $D(a,M)$. It is well known that $D(a,M) = \mathbf{B}^n(z,r)$ for some z and r.

2.2. Quasiregular mappings. A mapping $f\colon G \to \mathbb{R}^n$, $n \geq 2$, of a domain G in $\mathbb{R}^n$ is called quasiregular if f is in ACL^n, and there exists a constant K, $1 \leq K < \infty$ such that

$$|f'(x)|^n \leq K J_f(x), \quad |f'(x)| = \max_{|h|=1} |f'(x)h|,$$

a.e. in G, where $f'(x)$ is the formal derivative. The smallest $K \geq 1$ for which this inequality is true is called the outer dilatation of f and denoted by $K_O(f)$. If f is quasiregular, then the smallest $K \geq 1$ for which the inequality

$$J_f(x) \leq K l(f'(x))^n, \quad l(f'(x)) = \min_{|h|=1} |f'(x)h|,$$

holds a.e. in G is called the inner dilatation of f and denoted by $K_I(f)$. The maximal dilatation of f is the number $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leq K$, f is said to be K-quasiregular.

Let $f\colon G \to \mathbb{R}^n$ be a quasiregular mapping. We denote by $i(x,f)$ the infimum of $\sup_y \operatorname{card} f^{-1}(y) \cap U$ where U runs through the neighborhoods of x. The number $i(x,f)$ is called the local (topological) index of f at x.

Definition 2.3. *Let $f\colon \mathbf{H}^n \to \mathbb{R}^n$ be continuous and let $b \in \partial\mathbf{H}^n$. The mapping f is said to have*

1. *A sequential limit $\alpha \in \overline{\mathbb{R}}^n$ at 0 if there exists a sequence (b_k) in $\mathbf{H}^n$ with $b_k \to 0$ and $f(b_k) \to \alpha$.*
2. *A limit $\alpha \in \overline{\mathbb{R}}^n$ at 0 through a set E, if $0 \in \overline{E} \subset \mathbf{H}^n \cup \{0\}$ and $f(x) \to \alpha$ as $x \to 0$ and $x \in E$.*
3. *An approximate limit α at 0 if $\lim_{r\to 0} m\big((\mathbf{H}^n \setminus E_\varepsilon) \cap \mathbf{B}^n(r)\big) r^{-n} = 0$ for every $\varepsilon > 0$, where $E_\varepsilon = \{x \in \mathbf{H}^n : |f(x) - \alpha| < \varepsilon\}$.*
4. *An angular limit α at 0 if for each $\varphi \in (0, \pi/2)$, $f(x) \to \alpha$ when $x \to 0$ and $x \in C(\varphi) = \{z = (z_1, \ldots, z_n) \in \mathbf{H}^n : z_n > |z| \cos\varphi\}$.*

Remarks 2.4. (1) Let $f\colon \mathbf{B}^n \to \mathbb{R}^n$ be continuous, $b_k \in \mathbf{B}^n$, $b_k \to b \in \partial\mathbf{B}^n$, $f(b_k) \to \beta$ and $\beta \in \overline{\mathbb{R}}^n$. It is easy to see by continuity that there is an open set E, such that $b_k \in E$, $b \in \overline{E}$ and $f(b_k) \to \beta$, and if $x \to b$ for $x \in E$, then $f(x) \to \beta$.

(2) If $f\colon (X, d_1) \to (Y, d_2)$ is uniformly continuous, f has a sequential limit along (b_k) and if $r_k > r_{k+1} > 0$, $\lim_{k\to\infty} r_k = 0$, then f has a limit along the set $E = \bigcup_{k=1}^{\infty} B_k$, $B_k = \{z \in X : d_1(z, b_k) < r_k\}$.

(3) In (2) it is essential that $\lim_{k\to\infty} r_k = 0$. To see this consider the map $f\colon \mathbf{B}^2 \to \mathbf{B}^2 \setminus \{0\}$ defined by formula $f(z) = \exp(g(z))$, where $g(z) = -(1+z)/(1-z)$, $z \in \mathbf{B}^2$, fix $\beta \in \mathbf{B}^2 \setminus \{0\}$ and let $b_k \in \mathbf{B}^2$, $b_k \to 1$ with $f(b_k) = \beta$. Then $f\colon (\mathbf{B}^2, \rho) \to (\mathbf{B}^2, \rho)$ is uniformly continuous and there is $m > 0$ such that $fD(b_k, m)$ is a set independent of k.

Let $f\colon G \to \mathbb{R}^n$ be continuous. The set $C(f, b)$ of all sequential limits of f at a boundary point $b \in \partial G$ is called the cluster set of f at b. If $E \subset \partial G$ is nonempty, we denote $C(f, E) = \bigcup_{b\in E} C(f, b)$. It is clear that $C(f, b)$ is always a compact, nonempty subset of $\overline{fG}$. The mapping f is called boundary-preserving if $C(f, \partial G) \subset \partial fG$. Note that the closures here are taken with respect to $\overline{\mathbb{R}}^n$.

Theorem 2.5. *If $f\colon \mathbf{B}^n \to \mathbb{R}^n$ is K-quasiregular and $b_k \to b \in \partial\mathbf{B}^n$, $f(b_k) \to \beta$, $C(f, b) \subset \partial f\mathbf{B}^n$ and $E = \bigcup_{k=1}^{\infty} D(b_k, 1)$, then $\lim_{x\to b,\, x\in E} f(x) = \beta$.*

Proof. For $C(f, b) \neq \{\beta\}$, the claim follows from [11], Lemma 13.21 and Example 13.7.(1). If $C(f, b) = \{\beta\}$, then the claim is clearly true as $\lim_{x\to b,\, x\in\mathbf{B}^n} f(x) = \beta$. □

Corollary 2.6. *If $f\colon \mathbf{B}^n \to \mathbb{R}^n$ is a boundary preserving quasiregular mapping, $b_k \to b \in \partial\mathbf{B}^n$, $f(b_k) \to \beta$, then $\lim_{x\to b,\, x\in E} f(x) = \beta$, where $E = \bigcup_{k=1}^{\infty} D(b_k, 1)$.*

Definition 2.7. *Let $x \in \mathbb{R}^n$. If the set $A_x = \{r > 0 : S^{n-1}(x, r) \cap E \neq \emptyset\}$ is measurable we define the upper radial density of E at x by*

$$\operatorname{rad}\overline{\operatorname{dens}}(E, x) = \limsup_{r\to 0} m_1\big(A_x \cap (0, r)\big) r^{-1}.$$

Corollary 2.8. *If f is a boundary preserving quasiregular map, $f\colon \mathbf{B}^n \to f\mathbf{B}^n$ and $b_k \to e_1$, $f(b_k) \to \beta$, $b_k \in [0, e_1)$, then there is a set $E \subset [0, e_1)$ with $\operatorname{rad}\overline{\operatorname{dens}}(E, e_1) > 0$ such that $\lim_{x\to e_1,\, x\in E} f(x) = \beta$.*

Proof. By Corollary 2.6 we may apply Theorem 2.5, to find the set $E = \bigcup_{k=1}^{\infty} D(b_k, 1)$ such that $\lim_{x\to e_1,\, x\in E} f(x) = \beta$. From (2.1) it follows immediately that $\operatorname{rad}\overline{\operatorname{dens}}(E, e_1) > 0$. □

3. Measure densities and quasiregular mappings

In this section we define the upper and lower measure densities and study their relation to the boundary behavior of quasiregular mappings.

Definition 3.1. *Let $E \subset \mathbb{R}^n$ be a measurable set and $x \in \mathbb{R}^n$. The upper measure density of E at x is defined to be*

$$\theta^{n*}(E, x) = \limsup_{r\to 0} \frac{m\big(E \cap \overline{\mathbf{B}}^n(x, r)\big)}{\Omega_n r^n},$$

where $\Omega_n = m(\mathbf{B}^n)$, and the lower measure density $\theta^n_*(E, x)$ is the corresponding $\liminf$. *If $\theta^{n*}(E, x) = \theta^n_*(E, x)$, this common value is the measure density $\theta^n(E, x)$ of E at x.*

In order to prove the main result of this section, we need the following lemmas, first of which is a simple geometric observation.

Lemma 3.2. *Let $D_M = D(e_n, M)$ and $V_M = \mathbf{B}^n_+ \setminus D_M$. Then*
$$\frac{m(V_M)}{m(\mathbf{B}^n_+)} \le \frac{2\Omega_{n-1}}{\Omega_n}\frac{1}{\cosh M}.$$
In particular, $m(V_M)/m(\mathbf{B}^n_+) \to 0$ as $M \to \infty$.

Proof. Let $s_M = \mathrm{dist}(\partial D_M \cap S^{n-1}, \partial \mathbf{H}^n)$. By [11, 2.11.]
$$D(e_n, M) = \mathbf{B}^n\big((\cosh M)e_n, \sinh M\big).$$
By similar triangles we obtain the equality
$$s_M = 1/\cosh M.$$
Because $m(V_M) \le s_M\Omega_{n-1}$ and $m(\mathbf{B}^n_+) = \Omega_n/2$, we have
$$0 \le \frac{m(V_M)}{m(\mathbf{B}^n_+)} \le s_M\frac{2\Omega_{n-1}}{\Omega_n} = \frac{2\Omega_{n-1}}{\Omega_n}\frac{1}{\cosh M}.$$
□

The next lemma is a version of the Schwarz lemma for quasiregular mappings involving the local topological index $i(0, f)$. The applications of this result are based on the fact that the estimate improves when $i(0, f)$ grows. The basic geometric intuition behind this phenomenon can be seen by observing the behavior of the mapping $z \mapsto z^p$ in the complex plane when the exponent p grows to infinity.

Lemma 3.3. *Let $r \in (0, 1/2]$, and let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be a K-quasiregular mapping with $f(0) = 0$. Then*
$$|f(y)| \le \big(4r/3\big)^{\mu A};\ \ A = \beta^{1+2d\rho(y,0)} \in (0, 1)$$
for $y \in \mathbf{B}^n$, where $\mu = c_2 i(0, f)^{1/(n-1)}$, $c_2 > 0, d > 1$ are constants depending only on n, K and $\beta \in (0, 1)$ depends only on n, K and r.

Proof. By [4, Corollary 3.9.], $|f(x)| \le (4|x|/3)^\mu$ for $x \in \mathbf{B}^n(r)$ and by [8, Lemma 2.22.] we have $|f(y)| \le (4r/3)^{\mu A}$ for $y \in \mathbf{B}^n$. □

Lemma 3.4. *Let $f\colon \mathbf{H}^n \to \mathbf{B}^n$ be K-quasiregular with $f(z_k) = 0$, $z_k = |z_k|e_n$, $z_k \to 0$, c_2, d, β be as in Lemma 3.3 and let $\mu_k \equiv c_2 i(z_k, f)^{1/(n-1)} \to \infty$. If*
$$M_k = \frac{\log \mu_k}{4d \log \frac{1}{\beta}},$$
and $E = \bigcup_{k=1}^{\infty} D(z_k, M_k)$, then $f(x) \to 0$ as $x \to 0$, $x \in E$.

Proof. Let g_k be a Möbius transformation with $g_k(\mathbf{B}^n) = \mathbf{H}^n$ and $g_k(0) = z_k$. Let $r = 1/2$. Then by Lemma 3.3
$$|f \circ g_k(y)| \le \big(2/3\big)^{\mu_k A},$$
for $y \in \mathbf{B}^n$, where A is as in Lemma 3.3.

We need to find M_k such that for $\rho(y, z_k) \le M_k$,
$$\mu_k\beta^{1+2dM_k} \to \infty.$$

This holds for

$$\beta^{2dM_k} = \frac{1}{\sqrt{\mu_k}},$$

which is equivalent to

$$M_k = \frac{\log \mu_k}{4d \log \frac{1}{\beta}}.$$

□

Theorem 3.5. *Let* $f\colon \mathbf{H}^n \to \mathbf{B}^n$ *be a* K*-quasiregular mapping with* $f(z_k) = 0$, $z_k \in (0, e_n)$ *for* $|z_k| > |z_{k+1}| \to 0$ *and* $\mu_k \equiv c_2 i(z_k, f)^{1/(n-1)} \to \infty$. *Then there is a set* E *such that* $\{z_k\} \subset E$, $\lim_{x\to 0,\, x\in E} f(x) = 0$ *and* $\theta^{n*}(E,0) = \theta^n(\mathbf{H}^n, 0)$. *Furthermore,* $\operatorname{rad}\overline{\operatorname{dens}}(E,0) = 1$.

Proof. By Lemma 3.4, we may find M_k such that $\lim_{x\to 0,\, x\in E} f(x) = 0$ for $E = \bigcup_{k=1}^{\infty} D(z_k, M_k)$, and by Lemma 3.2

$$\limsup_{k\to\infty} \frac{m(E \cap \mathbf{B}^n_+(|z_k|))}{m(\mathbf{B}^n_+(|z_k|))} = 1.$$

It follows from (2.1) that $\operatorname{rad}\overline{\operatorname{dens}}(E,0) = 1$. □

4. Harnack condition and uniform Harnack condition

In this section we show that a quasiregular mapping has an angular limit at the origin, if it has a limit at the origin through a set with the complement of measure density zero in the upper half space $\mathbf{H}^n$. The proof is based on the Harnack inequality and related results from [9].

4.1. **Harnack's inequality.** Let G be a proper domain in $\mathbb{R}^n$, $n \geq 2$, and let $u\colon G \to \mathbb{R}$ be a continuous, nonnegative function. Then u is said to be a Harnack function if there are constants $\lambda \in (0,1)$ and $C_\lambda \geq 1$ such that the inequality

$$\max_{\overline{\mathbf{B}}^n(x,\lambda r)} u(z) \leq C_\lambda \min_{\overline{\mathbf{B}}^n(x,\lambda r)} u(z) \tag{4.2}$$

holds whenever $\mathbf{B}^n(x,r) \subset G$. A continuous, nonnegative function $u\colon G \to \mathbb{R}$ is said to satisfy a uniform Harnack inequality if it satisfies the Harnack inequality (4.2) for all $\lambda \in (0,1)$ and if $C_\lambda \to 1$ when $\lambda \to 0$.

4.3. **Modulus metric μ_G.** Let Γ be a path family in $\overline{\mathbb{R}}^n$. We denote by $\mathsf{M}(\Gamma)$ the conformal modulus of Γ. Let G be a proper subdomain of $\mathbb{R}^n$. For $x, y \in G$

$$\mu_G(x,y) = \inf_{C_{xy}} \mathsf{M}\big(\Delta(C_{xy}, \partial G; G)\big),$$

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0,1]$ and γ is a path with $\gamma(0) = x$ and $\gamma(1) = y$. Here $\Delta(C_{xy}, \partial G; G)$ is the family of all closed non-constant curvers joining C_{xy} and ∂G in G. The conformal invariant μ_G is called the modulus metric or the conformal metric of the domain G.

4.4. **Quasihyperbolic metric and metric j_G.** Let G a proper subdomain of $\mathbb{R}^n$. For $x \in G$ let $d(x) = \operatorname{dist}(x, \partial G) \in (0,\infty)$. We define

$$k_G(a,b) = \inf_{\gamma} \int_{\gamma} \frac{ds}{d(x)},$$

where γ runs through all rectifiable curves with $a, b \in \gamma$. It is well-known that k_G is a metric in G. The metric k_G is called the quasihyperbolic metric in G. From the definition it is clear that if G and G' are are domains with $G' \subset G$, then

$$k_G(x,y) \le k_{G'}(x,y); \qquad x, y \in G'. \tag{4.5}$$

Let

$$j_G(x,y) = \log\Big(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\Big)$$

for $x, y \in G$. It is well-known (see [11, 2.34.]) that $j_G(x,y)$ is also a metric in G.

A useful inequality ([3, Lemma 2.1]) is

$$j_G(x,y) \le k_G(x,y); \qquad x, y \in G. \tag{4.6}$$

A proper subdomain G of $\mathbb{R}^n$ is called uniform, if there exist a constant $A = A(G) \ge 1$ such that

$$k_G(x,y) \le A j_G(x,y); \qquad x, y \in G. \tag{4.7}$$

The unit ball $\mathbf{B}^n$ and the half space $\mathbf{H}^n$ are uniform domains with the constant $A = 2$ (see [1, 7.56] and [11, p.35]).

Lemma 4.8. *Let $f\colon \mathbf{H}^n \to \mathbf{B}^n$ be K-quasiregular and $\alpha \in S^{n-1}$. Then $u(x) = |f(x) - \alpha|$ satisfies the uniform Harnack inequality.*

Proof. By [11, Lemma 8.31.] and [11, 10.18.]

$$c_n j_{\mathbf{B}^n}\big(f(x), f(y)\big) \le \mu_{\mathbf{B}^n}\big(f(x), f(y)\big) \le K \mu_{\mathbf{H}^n}(x,y), \tag{4.9}$$

where c_n is a constant depending only on n. Now by uniformity of $\mathbf{B}^n$, (4.7), we have

$$k_{\mathbf{B}^n}\big(f(x), f(y)\big) \le 2 j_{\mathbf{B}^n}\big(f(x), f(y)\big). \tag{4.10}$$

Furthermore, $\mathbf{B}^n \subset \mathbb{R}^n \setminus \{\alpha\}$ and hence by (4.5),

$$k_{\mathbb{R}^n \setminus \{\alpha\}}\big(f(x), f(y)\big) \le k_{\mathbf{B}^n}\big(f(x), f(y)\big). \tag{4.11}$$

By [11, 3.5], the following inequality holds:

$$\left| \log \frac{|f(x) - \alpha|}{|f(y) - \alpha|} \right| \le k_{\mathbb{R}^n \setminus \{\alpha\}}\big(f(x), f(y)\big). \tag{4.12}$$

By combining (4.10), (4.11) and (4.12), we obtain

$$\left| \log \frac{|f(x) - \alpha|}{|f(y) - \alpha|} \right| \le 2 j_{\mathbf{B}^n}\big(f(x), f(y)\big),$$

and hence by (4.9)

$$u(x) \le \exp\Big(\frac{2K}{c_n} \mu_{\mathbf{H}^n}(x,y)\Big) u(y)$$

for all $x, y \in \mathbf{H}^n$. It follows that u satisfies the uniform Harnack inequality. $\square$

Theorem 4.13. [9, Theorem 6.13.] *Let $u\colon \mathbf{H}^n \to \mathbb{R}_+$ be a continuous function, $E \subset \mathbf{H}^n$ with $\theta^n(\mathbf{H}^n \setminus E, 0) = 0$, and suppose that $u(x) \to a$ as $x \to 0$ in the set E. Then u has an angular limit a at 0 if $u\colon \mathbf{H}^n \to \mathbb{R}_+$ satisfies the uniform Harnack inequality.*

Theorem 4.13 together with Lemma 4.8 immediately gives us the following result:

Theorem 4.14. *Let $f\colon \mathbf{H}^n \to \mathbf{B}^n$ be a quasiregular mapping, $\alpha \in S^{n-1}$ and $E \subset \mathbf{H}^n$ with $\theta^n(\mathbf{H}^n \setminus E, 0) = 0$. Suppose that $f(x) \to \alpha$ as $x \to 0$ in the set E. Then f has an angular limit α at 0.*

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