Mat-1.3602 Stochastic Analysis.

Exercise 18.3. 2008 Azmoodeh/Valkeila.

1. Assume that $X_n \overset{L^2(\mathbb{P})}{\to} X$ and $Y_n \overset{L^2(P)}{\to} Y$. Show that $(X_n, Y_n)_{L^2(\mathbb{P})} \to (X, Y)_{L^2(\mathbb{P})}$.
2. Let $\mathbb{F}$ be a history and let σ, τ be stopping times. Show that $\tau \wedge \sigma$ and $\tau \vee \sigma$ are stopping times.
3. Let τ be a stopping time and $\mathbb{F}$ is a history. Define F_τ by $F_\tau = \{A \in \mathcal{F} : A \cap \{t \leq n\} \in F_n\}$. Show that F_τ is a σ-algebra.
4. Let $\mathbb{F}$ be a history and let σ, τ be stopping times. Show that

$$F_{\tau \wedge \sigma} \subset F_\tau \subset F_{\tau \vee \sigma}.$$

5. Let $\mathcal{G}$ be a sub-σ-algebra and Y is a random variable. Show that if $|Y| \leq c$ a.s., then also $|\mathbb{E}(X|\mathcal{G})| \leq c$ a.s. Here c is a constant.
6. Let M be a square integrable martingale: $\mathbb{E}M_n^2 < \infty$ for all n. Show that $M_{n+k} - M_n \perp M_n$ and

$$\mathbb{E}\left((M_{n+k} - M_n)^2 |F_n\right) = \mathbb{E}\left(M_{n+k}^2 - M_n^2 |F_n\right).$$