Lectures on 22.4. and 24.4. 2008

5.2. **Path-wise stochastic integrals.** In some special cases one can define stochastic integrals as almost sure limits of Riemann-Stieltjes integrals, in other words as path-wise integrals. Of course we must restrict the class of processes which we can integrate in such a way.

Prologue. Let $f : C^1(\mathbb{R})$ and $t \mapsto X_t$, $X \in C^1(\mathbb{R})$ with the derivative X_t' The mean value theorem gives

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)X_s' ds \doteq \int_0^t f'(X_s)dX_s.$$

Itô formula without probability. Th path-wise results are due to Hans Föllmer.

Lemma 5.1. *Let W Brownian motion and π is a dyadic partition on $[0,t]$: $t_k = j2^{-n}t$, $j = 0, 1, \ldots, 2^n$. Then*

$$\sum_k (W_{t_k} - W_{t_{k-1}})^2 \to t \text{ a.s.} \tag{5.3}$$

Proof Let π be a dyadic partion on $[0,t]$ on $Y_n \doteq \sum_{k=1}^{2^n}(W_{t_k} - W_{t_{k-1}})^2$. Like in the proof of (3.2) we have

$$\mathrm{Var}(Y_n - t)2^n \cdot 2 \cdot t^2 2^{-2n} = t^2 2^{1-n}.$$

Let $\epsilon > 0$ and put $A_n = \{|Y_n - t| > \epsilon\}$. Using the Tsebysev inequality we obtain

$$\mathbb{P}(A_n) \leq \frac{1}{\epsilon^2}\mathrm{Var}(Y_n - t) = \frac{1}{\epsilon}t^2 2^{1-n}.,$$

This gives $\sum_n \mathbb{P}(A_n) < \infty$, and we can use Borel-Cantelli lemma to conclude that

$$\mathbb{P}(\limsup A_n) = 0 \Leftrightarrow Y_n \to t \quad \text{m.v.}$$

□

Remark 5.1. *One can improve the result of the previous lemma as follows: If the partitions satisfy $\pi^n \subset \pi^{n+1}$, then we have*

$$\sum_{t_k \in \pi^n} (W_{t_k} - W_{t_{k-1}})^2 \to t \text{ a.s.}$$

[Revuz, Yor, Proposition II.2.12].

In this subsection we assume that partitions of $[0,T]$ are dyadic.

Lemma 5.2. *Let $g : [0,T] \to \mathbb{R}$ be a continuous function. Then*

$$\int_0^T g(W_s)ds = \lim_{|\pi| \to 0} \sum_{t_k \in \pi} g(W_{k-1})(W_{t_k} - W_{t_{k-1}})^2 \tag{5.4}$$

Proof[7] Define $\mu^n := \sum_{t_k \in \pi^n}(W_{t_k} - W_{t_{k-1}})^2 \delta_{t_{k-1}}$. Here δ_{t_k} is the Dirac mass at t_k. Then we can write (5.3) as

$$\mu^n((0,u]) \to u$$

[7] The present proof is different to the proof given in the lecture.

when $u \leq t$, in other words the measures μ^n converge weakly to the Lebesgue measure on the interval $[0,t]$. If h is a continuous function on $[0,t]$, it is a bounded function on this interval, and so the weak convergence gives

$$\int_0^t h_s ds = \lim_n \int_0^t h_s \mu^n(ds) = \lim_{t_k \in \pi^n} h_{t_{k-1}} (W_{t_k} - W_{t_{k-1}})^2.$$

Apply this to the continuous function $h_s = g(W_s)$ to obtain (5.4). □.

Theorem 5.2. *Let $F \in C_2$ and π dyadic partition. Then we have the Itô formula*

$$F(W_T) = F(0) + \int_0^T F_x(W_s) dW_s + \frac{1}{2} \int_0^T F_{xx}(W_s) ds,$$

where the stochastic integral is a path-wise Riemann-Stieltjes limit

$$\int_0^T F_x(W_s) dW_s = \lim_{|\pi| \to 0} F_x(W_{t_{k-1}})(W_{t_k} - W_{t_{k-1}}). \tag{5.5}$$

Remark 5.2. *Let us compare this definition to the stochastic integral, which was defined as L^2- limit.*

- *The stochastic integral $H \circ W$ was defined in the case where the integrand H was a predictable process with $\mathbb{E} \int_0^T H_s^2 ds < \infty$. If H^n is a sequence of simple predictable processes with $\mathbb{E}_0^T (H_s - H_s^n)^2 ds \to 0$, then the integral $(H \circ W)_T$ is the $L^2(\mathbb{P})$ limit of random variables $(H^n \circ W)_T$. But L^2- convergence implies stochastic convergence and so*

$$\int_0^T H_s^n dW_s = \sum_k H_{t_{k-1}}^n (W_{t_k} - W_{t_{k-1}}) \xrightarrow{\mathbb{P}} (H \circ M)_T.$$

- *In the formula (5.5) the integrand has a a special form $F_x(W_s)$ and we pass to the limit using dyadic partitions. Now almost sure convergence implies stochastic convergence, and so the integral defined by (5.5) is the same as the stochastic integral defined earlier. In many applications it is important to have this kind of path-wise interpretation of stochastic integrals.*
- *The proof of (5.5) is based on Taylor expansion. Of course this is an alternative proof of the Itô- formula, and it is often proved in this way.*

Proof We start the proof with a simple observation: write

$$F(W_T) - F(0) = \sum_{k=1}^{2^n} \left(F(W_{t_k}) - F(W_{t_{k-1}}) \right).$$

Taylor expansion gives

$$F(W_{t_k}) - F(W_{t_{k-1}}) = F_x(W_{t_{k-1}})(W_{t_k} - W_{t_{k-1}}) + \frac{1}{2} F_{xx}(W_{t_k} - W_{t_{k-1}})^2 + R_{k,n},$$

where the correction term $R_{k,n}$ has the form

$$R_{k,n} = \frac{1}{2} \left(F_{xx}(\xi_k) - F_{xx}(W_{t_{k-1}}) \right) \left(W_{t_k} - W_{t_{k-1}} \right)^2$$

with $\xi_k \in (W_{t_{k-1}} \wedge W_{t_k}, W_{t_{k-1}} \vee W_{t_k})$.

We will show that

$$\sum_{k=1}^{2^n} R_{k,n} \to 0, \tag{5.6}$$

as $n \to \infty$. Let $\delta_n \doteq \max\{|W_{t_k} - W_{t_{k-1}}|, t_k \in \pi\}$. Brownian motion W has continuous paths, and hence $\delta_n \to 0$, as $n \to \infty$. Hence we can estimate further

$$|R_{k,n}| \leq \frac{1}{2} \max_{|x-y| \leq \delta_n} |F_{xx}(x) - F_{xx}(y)|(W_{t_k} - W_{t_{k-1}})^2 =: \epsilon_n (W_{t_k} - W_{t_{k-1}})^2.$$

The function F_{xx} is continuous, it is uniformly continuous on intervals $(-\delta_n, \delta_n)$. Hence $\epsilon_n \to 0$, as $n \to \infty$. We get now (5.6), since

$$|\sum_{k=1}^{2^n} R_{k,n}| \leq \sum_{k=1}^{2^n} |R_{k,n}| \leq \epsilon_n \sum_{k=1}^{2^n} (W_{t_k} - W_{t_{k-1}})^2 \to 0,$$

as $n \to \infty$. Lemma 5.2 gives

$$\sum_{k=1}^{2^n} F_{xx}(W_{t_{k-1}})(W_{t_k} - W_{t_{k-1}})^2 \to \int_0^T F_{xx}(W_s) ds.$$

This gives

$$\lim_n \sum_{k=1}^{2^n} F_x(W_{t_{k-1}})(W_{t_k} - W_{t_{k-1}}) = F(W_T) - F(0) - \frac{1}{2} \int_0^T F_{xx}(W_s) ds,$$

and we can interpret the limit as path-wise limit. □

5.3. **Girsanov theorem.** Recall that if X is a continuous semimartingale, then it has a unique representation

$$X = X_0 + M + A,$$

where M is a continuous local martingale and A is a continuous process with bounded variation on compacts, $M_0 = A_0 = 0$. Girsanov theorem explains how the martingale changes when we change the probability measure.

Stochastic exponential and stochastic logarithm. A useful tool to define change of measure is the stochastic exponential of a continuous martingale M.

Theorem 5.3. *Let M be a continuous local martingale, $M_0 = 0$. The unique solution to the equation*

$$Z_t = 1 + \int_0^t Z_s dM_s \tag{5.7}$$

is the stochastic exponential $\mathcal{E}(M)$ of M:

$$\mathcal{E}(M)_t = \exp(M_t - \frac{1}{2}\langle M, M\rangle_t).$$

For a positive martingale Z, $Z_0 = 1$, we can define the stochastic logarithm $\mathcal{L}(Z)$ by $\mathcal{L}(Z)_t = \int_0^t \frac{dZ_s}{Z_s}$. Then we have the following formulas $\mathcal{L}(\mathcal{E}(M)) = M$ and $\mathcal{E}(\mathcal{L}(Z)) = Z$.

Proof Itô formula gives that

$$\mathcal{E}(M)_t := e^{M_t - \frac{1}{2}\langle M,M\rangle_t}$$

is a solution to (5.7).
Uniqueness follows from the results on SDEs explained in the last lecture. The formulas with stochastic exponentials and stochastic logarithms are left as exercise. □
The proof of the following Yor's formula is left as an exercise:

$$\mathcal{E}(M)_t\mathcal{E}(N)_t = \mathcal{E}(M + N + [M,N])_t.$$

Before we continue, let us first discuss how to define measures with the help of Radon-Nikodym derivatives on filtered spaces. Assume that M is a continuous local martingale, $M_0 = 0$, and $\mathbb{E}_{\mathbb{P}}\mathcal{E}(M)_t = 1$ for all $t \geq 0$. This means that $\mathcal{E}(M)$ is a martingale [the details are in the extra set of exercises].
If $A \in F_t$, then we can define a measure Q_t on the sigma-algebra F_t by

$$Q_t(A) = \mathbb{E}_{\mathbb{P}}(\mathcal{E}(M)_t I_A).$$

Let show that this is consistent with respect to time: take $s < t$ and in $A \in F_s \subset F_t$. The martingale property of $\mathcal{E}(M)$ gives

$$Q_s(A) = \mathbb{E}_{\mathbb{P}}(\mathcal{E}(M)_s I_A) = \mathbb{E}_{\mathbb{P}}(\mathcal{E}(M)_t I_A) = Q_t(A).$$

This means that the restriction of the measure Q_t on the sigma-algebra F_s is Q_s: we can write this as $Q_t|F_s = Q_s$.

Theorem 5.4 (Girsanov)**.** *Let M be a local $(\mathbb{F},\mathbb{P})$- martingale and assume that $\mathbb{E}_{\mathbb{P}}\mathcal{E}(M)_t = 1$ for $t \geq 0$. Let X be another $(\mathbb{F},\mathbb{P})$- martingale. Then the process X is a $(\mathbb{F},Q)$- semimartingale and it has decomposition*

$$X = \tilde{X} + D,$$

where the process $\tilde{X} = X - D$ is a $(\mathbb{F},Q)$- martingale, $D = \langle X,M\rangle$ and the predictable angle bracket process of the martingale $X-D$ is $\langle X-D, X-D\rangle = \langle X,X\rangle$.

Proof
Let us assume that $|X_s| \leq K$, $\mathcal{E}(M)_s \leq K$ and $\mathcal{V}(\langle X,M\rangle)_t \leq K$, when $s \leq t$. With this assumption all the stochastic integrals in the proof below are martingales.
Let $A \in F_s$. Recall that $d\mathcal{E}(M)_t = \mathcal{E}(M)_d M_t$ and

$$\langle X,\mathcal{E}(M)\rangle_t = \int_0^t \mathcal{E}(M)_u d\langle X,M\rangle_u = \left(\mathcal{E}(M)\cdot\langle X,M\rangle\right)_t.$$

Let us now compute $\mathbb{E}_Q[X_t I_A]$:

$$\begin{aligned}
\mathbb{E}_Q[X_t I_A] &= \mathbb{E}_{\mathbb{P}}[\mathcal{E}(M)_t X_t I_A] \\
\text{integration by parts} &= \mathbb{E}_{\mathbb{P}}[I_A \left(\int_0^t \mathcal{E}(M)_u dX_u + \int_0^t X_u \mathcal{E}(M)_u dM_u + \langle X, \mathcal{E}(M) \rangle_t \right) \\
&= \mathbb{E}_{\mathbb{P}} \left(I_A \int_0^s \mathcal{E}(M)_u dX_u + \int_0^s X_u \mathcal{E}(M)_u dM_u + \langle X, \mathcal{E}(M) \rangle_t \right) \\
&= \mathbb{E}_{\mathbb{P}} \left(I_A \int_0^s \mathcal{E}(M)_u dX_u + \int_0^s X_u \mathcal{E}(M)_u dM_u + \langle X, \mathcal{E}(M) \rangle_s \right) \\
&\quad + \mathbb{E}_{\mathbb{P}}[I_A([X, \mathcal{E}(M)]_t - \langle X, \mathcal{E}(M) \rangle_s)] \\
\text{integration by parts} &= \mathbb{E}_{\mathbb{P}}[I_A \mathcal{E}(M)_s X_s] + \mathbb{E}_{\mathbb{P}}[I_A([X, \mathcal{E}(M)]_t - \langle X, \mathcal{E}(M) \rangle_s)],
\end{aligned}$$

where we have used the fact that the stochastic integrals are martingales.
Now $\mathbb{E}_{\mathbb{P}}[I_A \mathcal{E}(M)_s X_s] = \mathbb{E}_Q[I_A X_s]$ and integration by parts gives

$$\langle X, \mathcal{E}(M) \rangle = \mathcal{E}(M) \cdot \langle X, M \rangle = \langle X, M \rangle \mathcal{E}(M) - (\langle X, M \rangle \circ \mathcal{E}(M)).$$

Now we can combine these observations and we get

$$\mathbb{E}_Q[X_t I_A] = \mathbb{E}_Q[X_s I_A] + \mathbb{E}_Q[I_A(\langle X, M \rangle_t - \langle X, M \rangle_s)];$$

in other words the process $X - \langle X, M \rangle$ is a $(\mathbb{F}, Q)$- martingale.
We have proved this by assuming that $|X_s| \leq K$, $\mathcal{E}(M)_s \leq K$ and $\mathcal{V}_t(\langle X, M \rangle) \leq K$, when $s \leq t$.
Let

$$\tau_K = \inf\{u > 0 : |X_u| \geq K\} \wedge \inf\{u > 0 : \mathcal{E}(M)_u \geq K\} \wedge \inf\{u > 0 : \mathcal{V}_u(\langle X, M \rangle) \geq K\}.$$

Because the processes $X, \mathcal{E}(M), \langle X, M \rangle$ are continuous, and then we have that $\tau_K \to \infty$ as $K \to \infty$ and $\tau_K < \tau_{K+1}$. Let X be a local martingale η^n as the localizing sequence, M is a local martingale with σ^n as the localizing sequence. By the previous we get that $X - \langle X, M \rangle$ is a $(\mathbb{F}, Q)$- local martingale $\tau_n \wedge \eta^n \wedge \sigma^n$ as the localizing sequence.
Put $Y_t = (X_t - \langle X, M \rangle)_t$. Let $A \in F_s$ and we compute as above. Using the integration by parts formula [8]

$$dY_t^2 = 2Y_t dY_t + d\langle Y, Y \rangle_t = 2Y_t(dX_t - \langle X, M \rangle_t) + d\langle X, X \rangle_t;$$

hence the process Y^2 is a continuous semimartingale with respect to the measure $\mathbb{P}$ with $2Y \circ X$ as the martingale part and $\langle X, X \rangle - 2Y \cdot \langle X, M \rangle$ as the bounded variation part. Further

$$\langle \mathcal{E}(M), Y^2 \rangle = \langle \mathcal{E}(M), 2Y \circ X \rangle = 2(\mathcal{E}(M)Y) \cdot \langle X, M \rangle.$$

We can now compute:

$$\begin{aligned}
\mathbb{E}_Q(I_A Y_t^2) &= \mathbb{E}_{\mathbb{P}}(I_A Y_t^2 \mathcal{E}(M)_t) \\
\text{IP} &= \mathbb{E}_{\mathbb{P}}(I_A(2((\mathcal{E}(M)Y) \circ X)_t - 2(\mathcal{E}(M)Y) \cdot \langle X, M \rangle_t \\
&\quad + (Y^2 \cdot \mathcal{E}(M))_t + 2(\mathcal{E}(M)Y) \cdot \langle X, M \rangle_t + (\mathcal{E}(M) \cdot \langle X, X \rangle)_t) \\
&= \mathbb{E}_{\mathbb{P}}(I_A Y_s^2 \mathcal{E}(M)_s) \\
&\quad + \mathbb{E}_P[I_A((\mathcal{E}(M) \cdot \langle X, X \rangle)_t - ((\mathcal{E}(M) \cdot \langle X, X \rangle)_s)],
\end{aligned}$$

[8] In what follows we shall use the shorthand notation IP.

and this proves the second claim, since using IP we obtain

$$\mathbb{E}_{\mathbb{P}}[I_A((\mathcal{E}(M)\cdot\langle X,X\rangle)_t - ((\mathcal{E}(M)\cdot\langle X,X\rangle)_s)] = \mathbb{E}_Q[I_A(\langle X,X\rangle_t - \langle X,X\rangle_s)].$$

□

5.3.1. *Examples and applications.* Let X be a continuous process defined on $(\Omega, F, \mathbb{P})$. One can view the the process as a random variable with values in the space of continuous functions $C(\mathbb{R})$ or $C([0,T])$. We can define the metric in this space by putting: $f, g \in C([0,T])$ and then

$$||f-g||_\infty = \sup_{s\le T}|f(s)-g(s)|.$$

Let $\mathcal{B}$ be a Borel sigma-algebra generated by the open sets with respect to the norm $||\cdot||_\infty$. Now the process X induces a probability measure $\mathbb{P}^X$ in the space $C([0,T])$ for every $T>0$: if $B\in\mathcal{B}$, then

$$\mathbb{P}^X(B) = \mathbb{P}(\omega : X_.(\omega)\in B),$$

where we use the notation $X_.(\omega)$ for a fixed path of the process X. Sometimes it is convenient to work in the space $C([0,T])$ directly, and we identify ω with a function $x\in C([0,T])$: $x_t(\omega)=x_t$.

Example 5.1 (Brownian motion with a drift)**.** *Consider the following example. Let W be a Brownian motion defined on $(\Omega, F, \mathbb{P})$. As explained above, we can work in the canonical space of trajectories $C(\mathbb{R})$, and then Brownian motion induces a measure on $\mathbb{P}^W$ in the canonical space. Loosely speaking the measure $\mathbb{P}^W$ 'sees' only those functions, which have the properties of Brownian paths. For example, if we consider the following set $A\in\mathcal{B}$: $x\in A$ if and only if x is continuous and x has bounded variation on compacts. Then $\mathbb{P}^W(A)=0$.*
To get another measure in the canonical space consider the measure induced by the process $Y_t = W_t + at$, in other words Brownian motion with a drift. Call the corresponding measure by Q^a. From theorem 5.4 we obtain way to find the density:

- *In the Girsanov theorem the martingale X is Brownian motion W.*
- *With respect to the measure Q^a Brownian motion can be written as $W_t = \tilde{W}_t + at$, where $\tilde{W}$ is a $(\mathbb{F}, Q^a)$ martingale with $\langle\tilde{W},\tilde{W}\rangle_t = \langle W,W\rangle_t = t$, i.e. $\tilde{W}$ is a $(\mathbb{F}, Q^a)$ Brownian motion by Levy's characterization theorem.*
- *Take now $M = aW$, and then $\langle X,M\rangle_t = \langle W,aW\rangle_t = at$. Girsanovs theorem tells now that $\frac{dQ^a_t}{d\mathbb{P}^W_t} = \mathcal{E}(aW)_t$, and the process $\tilde{W}$ is a $(\mathbb{F}, Q^a)$ Brownian motion.*
- *Sometimes in the literature everything is written in the canonical space. Then $\frac{dQ^a_t}{d\mathbb{P}^W_t} = \mathcal{E}(ax)_t$, where x is a P^W Brownian motion.*

Example 5.2 (Statistical application)**.** *One can interpret the previous example also statistically. We know that $Q^a_T \prec\prec \mathbb{P}^W$, and because the exponential martingale $\mathcal{E}(ax)$ is strictly positive on the interval $[0,T]$, we also have that $\mathbb{P}^W \prec\prec Q^a$. Hence $Q^a_T \sim \mathbb{P}^W_T$, and if we observe a trajectory x on the interval $[0,T]$ we cannot say, weather we observe a Brownian motion or Brownian motion with a drift.*

The Radon-Nikodym derivative is in this case the likelihood ratio between the hypothesis $\mathbb{P}^W$ *and* Q^a*. So after observing the trajectory* x *on* $[0,T]$ *the likelihood ratio is*

$$\mathcal{E}(ax)_T = e^{ax_T - \frac{1}{2}a^2 T}.$$

The maximum likelihood estimator $\hat{a}_T$ *of the parameter* a *is the maximum of the likelihood, when the observation* x *is fixed. A direct computation gives* $\hat{a}_T = \frac{x_T}{T}$.

Next we apply some martingale techniques to analyze level crossing of Brownian motion, and Girsanov theorem to analyze level crossing of Brownian motion with a drift.

Lemma 5.3. *Let* W_t, $t \geq 0$ $(\mathbb{F}^W, \mathbb{P})$ *be a Brownian motion,* $a \in \mathbb{R}$ *and* $\tau_a = \inf\{u \geq 0 : W_u = a\}$. *Then* τ_a *is a stopping time,* $\mathbb{P}(\tau_a < \infty) = 1$ *and if* $\lambda > 0$*, then*

$$\mathbb{E}_{\mathbb{P}} e^{-\lambda \tau_\alpha} = e^{\sqrt{2\lambda}|a|}. \tag{5.8}$$

Proof We know τ_a is a stopping time. Put $M_t = \exp\{\sigma W_t - \frac{1}{2}\sigma^2 t\}$. We know that M is a martingale, and by Doobs stopping theorem

$$\mathbb{E}_{\mathbb{P}} M_{\tau_a \wedge t} = 1,$$

since $\tau_a \wedge t$ is a bounded stopping time.
Let $a > 0$; then

$$M_{\tau_a \wedge t} = \exp\{\sigma W_{\tau_a \wedge t} - \frac{1}{2}\sigma^2 \tau_a \wedge t\} \leq \exp\{\sigma a\}.$$

In the set $\tau_a < \infty$ we have the limit $\lim_{t\to\infty} M_{\tau_a \wedge t} = M_{\tau_a}$ and in the set $\tau_a = \infty$ we have $\lim_{t\to\infty} M_{\tau_a \wedge t} = 0$. DCT implies that $\mathbb{E}_{\mathbb{P}}(I_{\{\tau_a < \infty\}} M_{\tau_a}) = 1$, and using the fact that $W_{\tau_a} = a$ we get

$$\mathbb{E}_{\mathbb{P}} \left(I_{\{\tau_a < \infty\}} \exp\{-\frac{1}{2}\sigma^2 \tau_a\} \right) = e^{-\sigma a}.$$

Let now $\sigma \to 0$ and DCT gives again that

$$\mathbb{P}(\tau_a < \infty) = 1.$$

Moreover, the fact that $W_{\tau_a} = a$ gives the identity

$$\mathbb{E}_{\mathbb{P}} \exp\{-\frac{1}{2}\sigma^2 \tau_a\} = e^{-\sigma a}.$$

The formula (5.8) is obtained by choosing $\sigma = \sqrt{2\lambda}$.
If $a < 0$, then we can use the above argments together with the fact that $-W$ is also a Brownian motion, and in this case $\tau_a = \inf\{u : -W_u = -a\}$.
□
Next we show how one can use the previous result and Girsanov theorem to analyze the stopping time $\tau_a^b = \inf\{u : W_u + bt = a\}$. Here it is better to think that we work in the canonical space, and we look properties of the stopping time

$$\tau_a = \inf\{u : x(u) = a\}$$

from the point of the measure Q^a.

Theorem 5.5. *Let H be a bounded measurable functional from the space $(C_{[0,T]}, \mathcal{B})$ to the space $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$. Then*

$$\mathbb{E}_{Q^\mu} H = \mathbb{E}_{\mathbb{P}^W}(H\mathcal{E}(\mu W)_T), \tag{5.9}$$

where Q^μ is the distribution induced by the process $W_t + \mu t$ in the space $C_{[0,T]}, \mathcal{B})$ and $\mathbb{P}^W$ is the distribution induced by W_t.

Proof Let us assume that the functional H has the form $H(x_{t_1}, x_{t_2} - x_{t_1}, \dots, x_{t_n} - x_{t_{n-1}})$, when $x \in C_{[0,T]}$ and $t_k \in [0,T]$, $k = 1, \dots, n$. With respect to the measure Q^μ the vector $(x_{t_1}, x_{t_2} - x_{t_1}, \dots, x_{t_n} - x_{t_{n-1}})$ has the normal distribution, $x_{t_i} - x_{t_{i-1}} \sim N(\mu(t_i - t_{i-1}), t_i - t_{i-1})$ and the components of the vector are independent. Hence

$$\begin{aligned}
\mathbb{E}_Q(H) &= \int_{\mathbb{R}^n} H(y_1, \dots, y_n) \prod_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \\
&\quad \times e^{-\frac{1}{2(t_i - t_{i-1})}(y_i - \mu(t_i - t_{i-1}))^2} dy_1 \cdots dy_n \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \\
&\quad \times e^{-\frac{1}{2(t_i - t_{i-1})} y_i^2} e^{\sum_{i=1}^n \mu y_i - \frac{1}{2}\mu^2 \sum_{i=1}^n (t_i - t_{i-1})} dy_1 \cdots dy_n \\
&= \mathbb{E}_{\mathbb{P}}(H\mathcal{E}(\mu W)_T.
\end{aligned}$$

The general case follows from this, since the Q^μ distribution of every measurable functional can be approximated by distribtions of the functionals of the above type. □

Example 5.3. *Let us now study the stopping time τ_a, when we have distribution Q^μ. So $\tau_a = \inf\{t : x_t = a\}$, where $x \in C_{[0,T]}$. $H = e^{-\lambda\tau_a}$ is a bounded functional, and so*

$$\mathbb{E}_{Q^\mu} e^{-\lambda\tau_a \wedge T} = \mathbb{E}_{\mathbb{P}^W}\left(e^{-\lambda\tau_a \wedge T}\mathcal{E}(\mu x)_{\tau_a \wedge T}\right).$$

Note that $x_{\tau_a \wedge T} \le a$, and so

$$0 \le e^{-\lambda\tau_a \wedge T}\mathcal{E}(\mu x)_{\tau_a \wedge T} \le e^{|\mu a|}$$

and if $T \to \infty$, then

$$e^{-\lambda\tau_a \wedge T}\mathcal{E}(\mu x)_{\tau_a \wedge T} \to e^{-\lambda\tau_a}\mathcal{E}(\mu x)_{\tau_a} I_{\{\tau_a < \infty\}} = e^{\mu a} e^{-(\lambda + \frac{1}{2}\mu^2)\tau_a} I_{\{\tau_a < \infty\}}.$$

DCT theorem gives

$$\mathbb{E}_{Q^\mu}\left(e^{-\lambda\tau_a} I_{\{\tau_a < \infty\}}\right) = \mathbb{E}_{\mathbb{P}^W}\left(e^{\mu a} e^{-(\lambda + \frac{1}{2}\mu^2)\tau_a} I_{\{\tau_a < \infty\}}\right) = \exp(\mu a - |a|\sqrt{2\lambda + \mu^2}),$$

where the last equality follows from lemma 5.3. Let now $\lambda \to 0$ we get that

$$Q^\mu(\tau_a = \infty) = e^{\mu a - |\mu a|}.$$

This means that $Q^\mu(\tau_a = \infty) > 0$, if $\mu a \ne |\mu a|$.