2.4. **Martingale convergence theorem.** Martingale convergence theorem is a special type of theorem, since the convergence follows from structural properties of the sequence of random variables [2].

Theorem 2.8 (Martingale convergence theorem). *Let* $(X, \mathbb{F})$ *be a submartingale, which satisfies*

$$\sup_n \mathbb{E}X_n^+ < \infty. \tag{2.15}$$

Then the limit $\lim_n X_n = X_\infty$ *exists almost surely [and the limit is thus finite, almost surely]. Moreover, we have that* $X_\infty \in L^1$.

Remark 2.2. *We do not claim that the convergence in Theorem 2.8 is* $X_n \xrightarrow{L^1(\mathbb{P})} X_\infty$, *when* $n \to \infty$. *We shall give a counterexample to this claim in Example 2.6.*

Proof Let $u_n^X[a,b]$ be the number of upcrossings of the interval $[a,b]$ before the time n and put $u_\infty^X[a,b] = \lim_n u_n^X[a,b]$. Using the MCT and inequality (2.14) we will get that

$$\begin{aligned} \mathbb{E}u_\infty^X[a,b] &= \lim_n \mathbb{E}u_n^X[a,b] \\ &\le \frac{1}{b-a} \sup_n \mathbb{E}(X_n - a)^+ \\ &\le \frac{1}{b-a} \left(\sup_n \mathbb{E}X_n^+ + |a| \right), \end{aligned}$$

where we have used the fact that always we have $(x-a)^+ \le x^+ + |a|$. From this we will get that the random variable $u_\infty^X[a,b]$ is finite almost surely. Put

$$\Lambda_{a,b} = \{\limsup_n X_n \ge b, \liminf_n X_n \le a\}.$$

Since $u_\infty^X[a,b] < \infty$ almost surely, then $\mathbb{P}(\Lambda_{a,b}) = 0$. Let now a, b be rational numbers [3] and let $\Lambda = \cup_{a<b, a,b \in Q} \Lambda_{a,b}$. From above we know that $\mathbb{P}(\Lambda) = 0$. On the other hand we have that

$$\Lambda = \{\limsup_n X_n > \liminf_n X_n\}.$$

Hence we get that the limit $X_\infty = \lim_n X_n$ exists almost surely [but as indicated, the limit X_∞ is not necessarily finite].
Let us show that in fact $X_\infty \in \mathcal{L}^1$. X is a submartingale, and so $\mathbb{E}X_n \ge \mathbb{E}X_0$. Let us remark that

$$\begin{aligned} \mathbb{E}|X_n| &= \mathbb{E}X_n^+ + \mathbb{E}X_n^- \\ &= 2\mathbb{E}X_n^+ - \mathbb{E}X_n \\ &\le 2\mathbb{E}X_n^+ - \mathbb{E}X_0, \end{aligned}$$

[2] This result is somewhat analogous to the result from analysis, which says that a bounded monotone sequence always converges.

[3] In what follows we denote by Q the rational numbers.

and by Fatou's lemma and the estimate (2.15) give by using the estimate above that

$$\begin{aligned}\mathbb{E}|X_\infty| = \mathbb{E}\left(\lim_n |X_n|\right) &\le \liminf_n \mathbb{E}|X_n| \\ &\le 2\sup_n \mathbb{E}X_n^+ - \mathbb{E}X_0 < \infty;\end{aligned}$$

and so $X_\infty \in L^1$. □

Corollary 2.3. *If $(X, \mathbb{F})$ is non-negative supermartingale or if $(X, \mathbb{F})$ is a martingale bounded by avove (or below), then the limit $X_\infty = \lim_n X_n$ exists and $X_\infty \in L^1$.*

Proof If $(X, \mathbb{F})$ is non-negative supermartingale, then $(-X, \mathbb{F})$ is a submartingale satisfying (2.15). Hence non-negative supermartingale converges almost surely and and $X_\infty = \lim_n X_n$ is integrable.
Let $(X, \mathbb{F})$ be a martingale bounded below: $X_n \ge -c$ for some $c > 0$, when $n \ge 1$. The $Y_n = X_n + c$ is a non-negative martingale, and hence a supermartingale, and so $(Y_n, \mathbb{F})$ converges using the previous argument. This in turn implies that $(X, \mathbb{F})$ converges almost surely, and so $X_\infty = \lim_n X_n$ with $X_\infty \in L^1$.
Finally, let $(X, \mathbb{F})$ be a martingale bounded from above. Then $(-X, \mathbb{F})$ is a martingale bounded below, and after this observation everything should be clear. □

2.5. **Uniform integrable martingales.**

2.5.1. *Uniform integrability.*

Example 2.6. *Let ξ_k be i.i.d. random variables, $\mathbb{P}(\xi_k = 0) = \mathbb{P}(\xi_k = 2) = \frac{1}{2}$. We have that $\mathbb{E}\xi_k = 1$ and if $M_n = \prod_{k \le n} \xi_k$, then $(M, \mathbb{F}^M)$ is a martingale, $M_n \ge 0$ and $\sup_n \mathbb{E}M_n = 1$. Martingale convergence theorem applies, and we have that there exists almost sure limit $M_\infty = \lim_n M_n$. It is quite easy to see that $M_\infty = 0$ almost surely. Hence $\mathbb{E}M_\infty = 0 \ne 1 = EM_n$. This means that X_n does not converge to X_∞ in L^1.*

The next definition is essential for obtaining also the convergence in L^1.

Definition 2.6. *Let X_j, $j \in \mathcal{J}$ be a family of random variables. The family X_j, $j \in \mathcal{J}$ is <u>uniformly integrable</u>, if for all ϵ there exists $K > 0$ such that*

$$\int_{\{|X_j| > K\}} |X_j| d\mathbb{P} < \epsilon. \tag{2.16}$$

If X_j, $j \in \mathcal{J}$ is uniformly integrable family, then it is bounded in L^1: take $\epsilon = 1$ and $K > 0$ such that $\int_{\{|X_j|>K\}} |X_j| d\mathbb{P} < 1$. We obtain that $\mathbb{E}|X_j| < K + 1$ for all $j \in \mathcal{J}$.
On the other hand, a family of random variables, bounded in $L^1(\mathbb{P})$, then it is not necessarily uniformly integrable. Take

$$X_n = nI_{(0,1/n)} \text{ and } \mathbb{P} = Leb|(0, 1),$$

then $\mathbb{E}X_n = 1$ and for all $K > 0$

$$\int_{\{|X_n| > K\}} |X_n| d\mathbb{P} = 1,$$

when $n > K$.
But if the random variables are bounded in L^p, $p > 1$, then we have that they are unformly integrable as well.

Theorem 2.9. *Let the family X_j, $j \in \mathcal{J}$ be bounded in the space $L^p(\mathbb{P})$, $p > 1$. The the family X_j, $j \in \mathcal{J}$ is uniformly integrable.*

Proof Let $\epsilon > 0$, and let $q > 1$ be the Hlder conjugate of p and put $M = \sup_j \mathbb{E}|X_j|^p < \infty$. With th ehelp of Hlder inequality

$$\begin{aligned} \mathbb{E}\left(|X_j| I_{\{|X_j|>K\}}\right) &\le ||X_j||_p \left(\mathbb{P}(|X_j| > K)\right)^{\frac{1}{q}} \\ &\le \left(\mathbb{E}|X_j|^p\right)^{\frac{1}{p}} \left(\frac{\mathbb{E}|X_j|^p}{K^p}\right)^{\frac{1}{q}} = MK^{-\frac{p}{q}} < \epsilon, \end{aligned}$$

when $K > \left(\frac{M}{\epsilon}\right)^{\frac{q}{p}}$. □
To continue we need the following lemma:

Lemma 2.3. *Let $Y \in \mathbb{L}^1(\mathbb{P})$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mathbb{P}(A) < \delta \Rightarrow \mathbb{E}(|Y| I_A) < \epsilon$.*

Proof We give an indirect proof: there exists $\epsilon_0 > 0$ and A_n such that $\mathbb{P}(A_n) < 2^{-n}$ and $\mathbb{E}(|Y| I_{A_n}) > \epsilon_0$, when $n \ge 1$.
Put $H = \limsup_n A_n$. By construction of the sets A_n we have that $\sum_{k=1}^{\infty} \mathbb{P}(A_n) < \infty$, and so Borel-Cantelli lemma implies that $\mathbb{P}(H) = 0$, and so $\mathbb{E}(|Y| I_H) = 0$. On the other hand, by Fatou's lemma

$$\mathbb{E}(|Y| I_H) = \mathbb{E}(\limsup_n (|Y| I_{A_n})) \ge \limsup_n \mathbb{E}(|Y| I_{A_n}) \ge \epsilon_0;$$

but this is a contradiction, since wew should also have $\mathbb{E}(|Y| I_H) = 0$. Hence we are done. □

Theorem 2.10. *Let $X \in \mathbb{L}^1((\Omega, \mathcal{F}, \mathbb{P}))$ and put $Y^{\mathcal{G}} \doteq \mathbb{E}[X|\mathcal{G}]$, when $\mathcal{G} \subset \mathcal{F}$. Then the family $Y^{\mathcal{G}}$ is unformly integrable.*

Proof Using Lemma 2.3 given $\epsilon > 0$ we can pick $\delta > 0$ such that, if $\mathbb{P}(F) < \delta$, then $\mathbb{E}(|X| I_F) < \epsilon$. Moreover, from the fact that $X \in \mathbb{L}^1(\mathbb{P})$ it follows that we can find a constant $K > 0$ such that $\frac{\mathbb{E}(|X|)}{K} < \delta$.
Put $Y^{\mathcal{G}} = \mathbb{E}[X|\mathcal{G}]$, when $\mathcal{G} \subset \mathcal{F}$. From the Jensen inequality it follows that

$$|Y^{\mathcal{G}}| \le \mathbb{E}[|X||\mathcal{G}]. \tag{2.17}$$

From the estimate (2.17) we will get that $\mathbb{E}|Y^{\mathcal{G}}| \le \mathbb{E}|X|$ and further

$$K\mathbb{P}(|Y^{\mathcal{G}}| > K) \le \mathbb{E}|Y^{\mathcal{G}}| \le \mathbb{E}|X|. \tag{2.18}$$

With the help of the inequality (2.17) we finally get that

$$\int_{\{|Y^{\mathcal{G}}|>K\}} |Y^{\mathcal{G}}| d\mathbb{P} \le \int_{\{|Y^{\mathcal{G}}|>K\}} \mathbb{E}[|X||\mathcal{G}] = \int_{\{|Y^{\mathcal{G}}|>K\}} |X| d\mathbb{P},$$

where the last inequality follows from the defintion of conditional expectation. Now the claim follows from the Lemma 2.3, because $\mathbb{P}(|Y^{\mathcal{G}}| > K) < \delta$. □

Theorem 2.11. *Let $X_n, X \in \mathbb{L}^1(\mathbb{P})$. The following are equivalent:*

a) $\mathbb{E}|X_n - X| \to 0$, *as* $n \to \infty$.

b) *The family* X_n*,* $n \geq 1$ *is uniformly integrable and* $X_n \xrightarrow{\mathbb{P}} X$*.*

Proof First we prove the implication b) $\Rightarrow$ a): Let $K > 0$ and

$$\psi_K(x) = xI_{\{|x| \leq K\}} + KI_{\{x > K\}} - KI\{x < -K\}.$$

The family X_n, $n \geq 1$ is uniformly integrable, so there exists a constant $K > 0$ such, that

$$\mathbb{E}|\psi_K(X_n) - X_n| = |\int_{\{|X_n| > K\}} X_n d\mathbb{P}| \leq \int_{\{|X_n| > K\}} |X_n| d\mathbb{P} \leq \frac{\epsilon}{3}. \tag{2.19}$$

Clearly $|\psi_K(x) - \psi_K(y)| \leq |x - y|$ and this in turn implies that the map $x \mapsto \psi_K(X)$ is continuous. Continuous mappings preserve convergence in probability, and so $\psi_K(X_n) \xrightarrow{\mathbb{P}} \psi_K(X)$. DCT theorem implies, that in fact this convergergence in in L^1, and hence there exists n_0 such that if $n \geq n_0$, then

$$\mathbb{E}|\psi_K(X_n) - \psi_K(X)| < \frac{\epsilon}{3}. \tag{2.20}$$

We can now collect the estimates (2.19) and (2.20) and obtain

$$\begin{aligned} \mathbb{E}|X_n - X| &\leq \mathbb{E}|X_n - \psi_K(X_n)| + \mathbb{E}|\psi_K(X_n) - \psi_K(X)| \\ &+ \mathbb{E}|\psi_K(X) - X| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Hence we have a).
Convesely, assume a) : $\mathbb{E}|X_n - X| \to 0$, as $n \to \infty$. Because $\mathbb{L}^1$ convergence implies stochastic convergence, then $X_n \xrightarrow{\mathbb{P}} X$.
Next we will show that the family $X_n, n \geq 1$ is uniformly integrable. Let $\epsilon > 0$ and n_0 such that for $n \geq n_0$ we have $\mathbb{E}|X_n - X| < \frac{\epsilon}{2}$. Lemma 2.3 gives the following: there exists $\delta > 0$ such that if $\mathbb{P}(A) < \delta$, then

$$\mathbb{E}|X_k I_A| < \epsilon \text{ ja } \mathbb{E}|X I_A| < \frac{\epsilon}{2}, \tag{2.21}$$

when $k = 1, \ldots, n_0$ [formally the constant δ depends from the index k, but we can take $\delta = \min\{\delta_k : k \leq n_0\}$, and $\delta > 0$, since the minimum is taken over a finite collection of positive numbers].
The family X_n, $n \geq 1$ is bounded in $\mathbb{L}^1(\mathbb{P})$, and hence there exists $K > 0$ such that we have

$$\frac{\sup_n \mathbb{E}|X_n|}{K} < \delta. \tag{2.22}$$

From this inequality we obtain that $\mathbb{P}(|X_n| > K) < \delta$, $n \geq 1$, so if $k \leq n_0$, then $\mathbb{E}|X_k I_{\{|X_k| > K\}}| < \epsilon$ and if $k > n_0$, then

$$\mathbb{E}|X_k I_{\{|X_k| > K\}}| \leq \mathbb{E}|X_k - X| + \mathbb{E}|X I_{\{X_k\}}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence the family X_n, $n \geq 1$ is uniformly integrable and we have proved b).
□
As a corollary we get the very useful summary:

Corollary 2.4. *Let* X_n*,* $n \geq 1$ *be non-negative random variables which satisfy* $X_n \xrightarrow{\mathbb{P}} X$*,* $X \in \mathbb{L}^1(\mathbb{P})$*. Then the following are equivalent:*

a) $\mathbb{E}|X_n - X| \to 0$*, as* $n \to \infty$*.*
b) *The family* X_n*,* $n \geq 1$ *is uniformly integrable.*

c) $\mathbb{E}X_n \to \mathbb{E}X$, *as* $n \to \infty$.

Proof The claim a) $\Leftrightarrow$ b) is in fact Theorem 2.11.
Assume that a) holds. Because $|\mathbb{E}X_n - \mathbb{E}X| \leq \mathbb{E}|X_n - X|$, then we have c).
Coversely, assume c). We have the identity

$$|X_n - X| = X_n - X + 2(X_n - X)^-. \tag{2.23}$$

By assumption $(X_n - X)^- \xrightarrow{\mathbb{P}} 0$ and since X_n, X are non-negative, we get the estimate $(X_n - X)^- \leq X$. Now we can use the dominated convergence theorem to get

$$\mathbb{E}(X_n - X)^- \to 0 \text{ as } n \to \infty.$$

So the claim a) follows from this fact, and identity (2.23). □

2.5.2. *Uniform integrable martingales.*

Definition 2.7. *$(M_n, F_n)_{n\geq 1}$ is an uniformly integrable martingale, if $(M_n, F_n)_{n\geq 1}$ is a martingale, and the family M_n, $n \geq 1$ is uniformly integrable.*

Let $(M_n, F_n)_{n\geq 1}$ be uniformly integrable martingale. Uniform integrability implies that the family $M_n, n \geq 1$ is bounded in $L^1(\mathbb{P})$, and so $\sup \mathbb{E}|M_n| < \infty$. We can use the martingale conergence theorem to conclude that we have the limit

$$M_\infty = \lim_n M_n \mathbb{P} - \text{ a.s.};$$

moreover, by Theorem 2.11 we get that $\mathbb{E}|M_\infty - M_n| \to 0$, as $n \to \infty$.
We claim that in fact

$$M_n = \mathbb{E}[M_\infty | F_n]. \tag{2.24}$$

By the martingale property we get, when $F \in F_n$ and $r > n$:

$$\int_F M_n d\mathbb{P} = \int_F M_r d\mathbb{P}. \tag{2.25}$$

On the other hand

$$\begin{aligned} |\int_F M_r d\mathbb{P} - \int_F M_\infty d\mathbb{P} &\leq \int_F |M_r - M_\infty| d\mathbb{P} \\ &\leq \mathbb{E}|M_r - M_\infty| \to 0, \text{ kun } r \to \infty. \end{aligned}$$

This shows that we can replace in the equation (2.25) the random variable M_r by the random variable M_∞. From this we obtain the claim (2.24).
As a summary of uniform integrable martingales we obtain

Theorem 2.12. *Assume that $(M_n, F_n)_{n\geq 1}$ is a uniformly integrable martingale. Then there exists an integrable random variable M_∞, which satisfies*

$$M_\infty = \lim_n M_n \mathbb{P} - \text{ a.s. and in } \mathbb{L}^1(\mathbb{P}).$$

Moreover, we have the representation (2.24) .

Corollary 2.5. *Let (M_n, F_n) $n \geq 1$ be an uniformly integrable martingale and τ is a stopping time. Then*

$$M_\tau = \mathbb{E}[M_\infty | F_\tau]. \tag{2.26}$$

To verify this is left as an exercise.

2.6. **Change of measure.**

2.6.1. *Radon-Nikodym-thoerem.*

Definition 2.8. *probability measure Q is absolutely continuous with respect to the probability measure $\mathbb{P}$, if $\mathbb{P}(A) = 0 \Rightarrow Q(A) = 0$. Notation: $Q \prec\prec \mathbb{P}$. The measures Q ja $\mathbb{P}$ are equivalent if $Q \prec\prec \mathbb{P}$ ja $\mathbb{P} \prec\prec Q$. Notation: $Q \sim \mathbb{P}$.*

Example 2.7. *Let $X \geq 0$ be a random variable with $\mathbb{E}_{\mathbb{P}} X = 1$. Then one can define a new probability measure Q by putting*

$$Q(A) = \int_A X d\mathbb{P}.$$

We know that Q is a probability measure and that $Q \prec\prec \mathbb{P}$ [after the exercise III, at least].

Theorem 2.13 (Radon-Nikodym-theorem)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{P}, Q$ probability measures on it. Let $Q \prec\prec \mathbb{P}$. Then there exists a random variable $Y \geq 0$ such that*

$$Q(A) = \int_A Y d\mathbb{P}.$$

We say that Y is the Radon-Nikodym derivative of Q with respect to the measure $\mathbb{P}$. Notation: $Y =: \frac{dQ}{d\mathbb{P}}$.

Proof[4] We prove the theorem in the case when $\mathcal{F}$ is separable [or countably generated]: there exists sets $A_1, A_2, \ldots$ such that $\mathcal{F} = \sigma\{A_i : i \geq 1\}$[5].
Let $\mathcal{F}_n = \sigma\{A_1, \ldots, A_n\}$; we know that if $B \in \mathcal{F}_n$, then $B = \cup_{i=1}^k B_k$, where the sets B_k have the form $B_k = \cap_{i=1}^p C_i$, and $C_i = A_j$ or $C_i = A_j^c$ for some $A_j \in \{A_1, \ldots, A_n\}$. We say that a set $D \in \mathcal{F}_n$ is an atom, if $G \subset D$ implies that $G = D$ or $G = \emptyset$. It is easy to check that the atoms $A_{n,1}, \ldots, A_{n,r(n)}$ of the sigma-algebra $\mathcal{F}_n$ define a partition of the set Ω: $\Omega = \sum_{k=1}^{r(n)} A_{n,k}$ and if $G \in \mathcal{F}_n$, then $G = \sum_{j=1}^p A_{n,k_j}$. Denote the atoms of $\mathcal{F}_n$ by $A_{n,j}$, $j = 1, \ldots, r(n)$.
Clearly $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Put $Y_n(\omega) = 0$, when $\omega \in A_{n,j}$ and $\mathbb{P}(A_{n,j}) = 0$ [then also $Q(A_{n,j}) = 0$, since $Q \prec\prec \mathbb{P}$] and otherwise we put

$$Y_n(\omega) = \frac{Q(A_{n,j})}{\mathbb{P}(A_{n,j})}.$$

We will show that $(Y_n, \mathcal{F}_n)$, $n \geq 1$ is $\mathbb{P}$- martingaale. Let $F \in \mathcal{F}_n$. Because also $F \in \mathcal{F}_{n+1}$, then there exists atoms A_{n+1,j_k} such that

$$F = \cup_{k=1}^p A_{n+1,j_k}.$$

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[5]One can show that Borel sets on real line is separable, but Lebesgue measurable sets are not separable.

Now we can compute

$$Q(F) = \sum_{k=1}^{p} Q(A_{n+1,j_k}) \quad = \quad \sum_{k=1}^{p} \frac{Q(A_{n+1,j_k})}{\mathbb{P}(A_{n+1,j_k})}\mathbb{P}(A_{n+1,j_k})$$
$$= \int_F Y_{n+1}\mathbb{P}.$$

And with a similar computation one can show that $Q(F) = \int_F Y_n d\mathbb{P}$; now we have shown that

$$\int_F Y_n d\mathbb{P} = \int_F Y_{n+1} d\mathbb{P},$$

so the sequence $(Y_n, \mathcal{F}_n)$, $n \geq 1$ is a martingale.
On the other hand, $Y_n \geq 0$, $\mathbb{E}_{\mathbb{P}} Y_n = 1$, and so the martingale $(Y_n, \mathcal{F}_n)_{n\geq 1}$ converges almost surely and we have a limit $Y_\infty = \lim_n Y_n$ $\mathbb{P}$- a.s.
We will show next that $(Y_n, \mathcal{F}_n, \mathbb{P})_{n\geq 1}$ is an uniformly integrable martingale. The following is an exercise:

Lemma 2.4. *Let $Q \prec\prec \mathbb{P}$: then for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mathbb{P}(F) < \delta \Rightarrow Q(F) < \epsilon$.*

Now we prove unifrm integrability. Let $\epsilon > 0$; then we can find $\delta > 0$ such that $\mathbb{P}(F) < \delta \Rightarrow Q(F) < \epsilon$. Pick a constant $K > 0$ such that

$$\frac{1}{K} = \frac{Q(\Omega)}{K} < \delta.$$

Then

$$\mathbb{P}(Y_n > K) \leq \frac{\mathbb{E}_{\mathbb{P}} Y_n}{K} = \frac{Q(\Omega)}{K} < \delta;$$

and by using Lemma 2.4 we finaly get

$$\int_{\{Y_n > K\}} Y_n d\mathbb{P} = Q(Y_n > K) < \epsilon.$$

Hence the family Y_n, $n \geq 1$ is uniformly integrable with respect to the measure $\mathbb{P}$, and $Y_\infty = \lim_n Y_n$ also in $L^1(\mathbb{P})$. Moreover $\mathbb{E}_{\mathbb{P}} Y_\infty = 1$.
Define a probability measure R by

$$R(A) = \int_A Y_\infty d\mathbb{P}, \text{ when } A \in \mathcal{F}.$$

We claim that $R = Q$.
We know that $Y_n = \mathbb{E}_{\mathbb{P}}[Y_\infty | \mathcal{F}_n]$, so if $A \in \mathcal{F}_n$, the by definition of the conditional expectation

$$Q(A) = \int_A Y_n d\mathbb{P} = \int_A Y_\infty d\mathbb{P} = R(A).$$

hence the probability meassures R ja Q are the same on $\mathcal{H} = \cup_n \mathcal{F}_n$. But the family $\mathcal{H}$ is a monotone class, and by extension theorem we get that the measures Q and R are the same on $\mathcal{F}$. □

Remark 2.3.

- *The Radon-Nikodym derivate $Y = \frac{dQ}{d\mathbb{P}}$ is unique almost surely.*
- *One can show that Theorem 2.13 holds for arbitrary $\mathcal{F}$*

27.3. 2008 [with some leftover on Tu 1.4. 2008]