2.1.2. *Martingale transforms and predictability.*

Definition 2.3. *Let* $\mathbb{F} = (\mathcal{F}_k)_{k\geq 0}$ *be a history. Stochastic process* $C = (C_k)_{k\geq 1}$ *is predictable, if* $C_k \in \mathcal{F}_{k-1}\ \forall k \geq 1$.

Definition 2.4. *Let* C *and* X *be stochastic processes. The process* $C \circ X$ *is martingale transform, where*

$$(C \circ X)_n \doteq \sum_{k=1}^{n} C_k (X_k - X_{k-1}) = \sum_{k=1}^{n} C_k \Delta X_k,$$

when $n \geq 1$ *and* $(C \circ X)_0 = X_0$.

The next theorem justifies the terminology.

Theorem 2.1. *Let* $\mathbb{F}$ *be a history, the process* X *satisfies* $X \in \mathbb{F}$ *and* C *is a predictable process.*

- *If in addition* $0 \leq C_n(\omega) \leq K$ *and* X *is a supermartingale, then* $Y \doteq (C \circ X)$ *is a supermartingale.*
- *If in addition* $|C_n(\omega)| \leq K$ *and* X *is a martingale, then* $Y \doteq (C \circ X)$ *is a martingale.*

The proof is left as an exercise.

2.1.3. *Stopping times and processes.* let $\mathbb{F}$ be a history, X is $\mathbb{F}$- adapted process and τ is a stopping time. Define the random variable X_τ by

$$X_\tau = \sum_{k=0}^{\infty} X_k I_{\{\tau = k\}}.$$

Let us define $X_\infty I_{\{\tau=\infty\}} = 0$, if there is no limit $\lim_n X_n$. Since

$$\{X_\tau \in B\} = \cup_{n=0}^{\infty} \left(\{X_n \in B\} \cap \{\tau = n\}\right) \in \mathcal{F},$$

when $B \in \mathbb{B}_{\mathbb{R}}$, then X_τ is well defined random variable.

Definition 2.5. *Let* $\mathbb{F}$ *be a history and* τ *is a stopping time. The stopped* σ*-algebra is*

$$F_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq k\} \in F_k, k \geq 1\}.$$

Theorem 2.2. *Let* $\mathbb{F}$ *be a history,* τ *is a stopping time and* X $\mathbb{F}$*- adapted process. Then* $X_\tau \in \mathcal{F}_\tau$.

Proof Let B be a Borel set and consider again

$$\{X_\tau \in B\} = \cup_{n=0}^{\infty} \left(\{X_n \in B\} \cap \{\tau = n\}\right).$$

By definition, $\{X_n \in B\} \cap \{\tau = n\} \in F_\tau$, and the claim follows. □

Stopped σ-agebras have the property:

$$\sigma \leq \tau \Rightarrow F_\sigma \subset F_\tau \tag{2.1}$$

and if $\mathbb{P}(\tau \leq M) = 1$ and $X = (X_k)_{k\geq 0}$ is a martingale, then

$$\mathbb{E}X_\tau = \mathbb{E}X_0. \tag{2.2}$$

[This follows from Theorem 2.3 proved a bit later.]

Let X be a stochastic process, $\mathbb{F}$ a history, and τ a stopping time. Assume that $X \in \mathbb{F}$. Define the stopped process X^τ and the stopped history $\mathbb{F}^\tau$ by:

$$X_n^\tau = X_{\tau\wedge n} \quad \text{and} \quad F_n^\tau = F_{\tau\wedge n}.$$

We have, for any Borel set B, that

$$\{X_n^\tau \in B\} = \sum_{k=0}^{n} \{X_k \in B\} \cap \{\tau = k\} + \{X_n \in B\} \cap \{\tau > n\},$$

and so $X_n^\tau \in F_n^\tau$. We can write this as $X^\tau \in \mathbb{F}^\tau$. Next we study how stopping affects martingale properties of the process X.

Lemma 2.1. *Let $(X, \mathbb{F})$ be a (super)martingale and τ is a stopping time. Then $(X^\tau, \mathbb{F})$ is a (super)martingale.*

Proof Put $C_k = I_{\{k\leq\tau\}}$, when $k \geq 1$. The process C is predictable: $\{C_k = 0\} = \{\tau \leq k-1\} \in F_{k-1}$; note also that

$$X_n^\tau = \sum_{k=1}^{n} C_k(X_k - X_{k-1}),$$

and so X^τ is a (super)martingale by Theorem 2.1. □

Theorem 2.3 (Doob)**.** *Let X be a martingale, which is defined on a stochastic basis $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. Let σ, τ be bounded stopping times, which satisfy $\sigma \leq \tau$. Then*

$$\mathbb{E}[X_\tau | \mathcal{F}_\sigma] = X_\sigma. \tag{2.3}$$

Proof By assumption we have that $\sigma \leq \tau \leq M$. We must show that for $A \in F_\sigma$ we have

$$\int_A X_\sigma d\mathbb{P} = \int_A X_\tau d\mathbb{P}.$$

For this it is enough to show that

$$\int_{A\cap\{\sigma=k\}} X_\sigma d\mathbb{P} = \int_{A\cap\{\sigma=k\}} X_\tau d\mathbb{P}$$

when $1 \leq k \leq M$. Put $B = A \cap \{\sigma = k\}$, and now we get, using recursion and the martingale property of X:

$$\begin{aligned}
\int_B X_\sigma d\mathbb{P} &= \int_B X_k d\mathbb{P} = \int_{B\cap\{\tau=k\}} X_k d\mathbb{P} + \int_{B\cap\{\tau>k\}} X_k d\mathbb{P} \\
&= \int_{B\cap\{\tau=k\}} X_k d\mathbb{P} + \int_{B\cap\{\tau>k\}} X_{k+1} d\mathbb{P} \\
&= \int_{B\cap\{\tau=k\}} X_\tau d\mathbb{P} + \int_{B\cap\{\tau=k+1\}} X_\tau d\mathbb{P} + \int_{B\cap\{\tau>k+1\}} X_{k+1} d\mathbb{P} \\
&\cdots \\
&= \int_{B\cap\{k\leq\tau\leq M\}} X_\tau d\mathbb{P} = \int_B X_\tau d\mathbb{P}.
\end{aligned}$$

This proves the claim (2.3). □.

Remark 2.1.
- *A more general version of Doob's stopping theorem goes as follows: X is a martingale on $(\Omega, \mathbb{F}, F, \mathbb{P})$ and σ, τ are two stopping times with the property that $X_\sigma, X_\tau \in L^1(\mathbb{P})$. If in addition*
$$\liminf_n \int_{\{\tau > n\}} |X_\tau| d\mathbb{P} = 0,$$
then
$$\mathbb{E}[X_\tau | F_\sigma] 1_{\{\tau \geq \sigma\}} = X_\sigma 1_{\{\tau \geq \sigma\}}.$$
- *As a corollary of Doob's theorem we obtain that X^τ is a $\mathbb{F}^\tau$ martingale.*

Example 2.5. *Let ξ_k be independent Bernoulli variables with $\mathbb{P}(\xi_k = 1) = p = 1 - \mathbb{P}(\xi_k = -1)$, when $k \geq 1$. Let $X_0 = 0$ and $X_n = \sum_{k=0}^n \xi_k$. Put $F_n = F_n^X = \sigma(\xi_1, \ldots, \xi_n)$.*
The player can put any amount V_n in the game on the n^{th} round. How the player chooses V_n can depend on the previous results $\xi_1, \ldots, \xi_{n-1}$ of the game. Hence V_n is measurable with respect to F_{n-1}, or in other words V is predictable with respect to $\mathbb{F}^X$. We interpret $\xi_n = 1$ as the win: the player receives V_n from the bank, if the outcome of the n^{th} round is $\xi_n = -1$, then the player pays V_n to the bank. The gains process G is the following:
$$G_n = \sum_{k=1}^n V_k \xi_k = \sum_{k=1}^n V_k \Delta X_k = (V \circ X)_n.$$
If $p = \frac{1}{2}$, the process X is a martingale, and the gains process G is also a martingale, provided that G_n is integrable.
The martingale strategy goes as follows: bet $V_1 = 1$, and if $\xi_1 = 1$, then stop. If $\xi_1 = -1$, then bet $V_2 = 2$. And more generally, if $\xi_1 = -1, \ldots, \xi_{n-1} = -1$ then bet $V_n = 2^n$, and if $\xi_n = 1$, then stop.
Put $\tau = \inf\{k : \xi_k = 1\}$. τ is a stopping time. Clearly, if $\tau = n$, we have
$$G_n = 2^n - \sum_{k=1}^n 2^{k-1} = 2^n - 2^n + 1 = 1.$$
Hence $G_\tau = 1$.
- *Good news for the gambler: $\mathbb{P}(\tau < \infty) = 1$ [even for any $p > 0$].*
- *Bad news for the gambler: $\mathbb{E}|G_\tau| = \infty$ for $p \leq \frac{1}{2}$ [but $\mathbb{E}|G_\tau| = \frac{2}{2(2p-1)} < \infty$ for $p > \frac{1}{2}$.]*

2.2. Some martingale inequalities.

2.2.1. *Doob maximal inequality.* Let X be a process on $(\Omega, \mathbb{F}, F, \mathbb{P})$ and put
$$X_n^* = \max_{k \leq n} |X_k|;$$
by assumption $X_k \in F_k \subset F_n$, when $k \leq n$, then we get $X_n^* \in F_n$. Clearly the process X is increasing and non-negative. The process X^* is the *maximal process* .of X.
With the Markov inequality we have for all $c > 0$

$$c\mathbb{P}(X_n^* \geq c) \leq \mathbb{E}(X_n^* I_{\{X_n^* \geq c\}}) \leq \mathbb{E}X_n^*. \tag{2.4}$$

The next theorem tells that if $(X_n, F_n)_{n\geq 0}$ is a non-negative submartingale, then on the right hand side of (2.4) one can replace the value X_n^* of the maximal process by the value of submartingale X_n!

Theorem 2.4 (Doob maximal inequality)**.** *Let* $(Z, \mathbb{F})$ *be a non-negative submartingale. Then for all* $c > 0$

$$c\mathbb{P}(Z_n^* \geq c) \leq \mathbb{E}(Z_n I_{\{Z_n^* \geq c\}}) \leq \mathbb{E}Z_n. \tag{2.5}$$

Proof Put $G = \{Z_n^* \geq c\}$. Let us define recursively $G_0 = \{Z_0 \geq c\}$ and for $k \geq 1$

$$G_k = \{Z_0 < c\} \cap \{Z_1 < c\} \cap \cdots \cap \{Z_k \geq c\}.$$

By definition $G_k \in F_k$ and if $k \neq l$, then $G_k \cap G_l = \emptyset$. Moreover, we have that $G = \cup_{k=0}^n G_k$.
After these preparations we can prove the inequality (2.5). because Z is a submartingale, then for all $0 \leq k \leq n$ it holds

$$\mathbb{E}(Z_n I_{G_k}) = \int_{G_k} Z_n d\mathbb{P} \geq \int_{G_k} Z_k d\mathbb{P} \geq c\mathbb{P}(G_k); \tag{2.6}$$

since the events G_k are disjoint and their union is G:

$$\begin{aligned} c\mathbb{P}(X_n^* \geq c) &= c\mathbb{P}(G) = c\sum_{k=0}^n \mathbb{P}(G_k) \\ \text{estimate}(2.6) &\leq \sum_{k=0}^n \int_{G_k} Z_n d\mathbb{P} = \mathbb{E}(Z_n I_G). \end{aligned}$$

This proves the inequality (2.5). □
Let $(M, \mathbb{F})$ be a martingale. Then $(|M|, \mathbb{F})$ is a nonnegative submartingale, and we have

Corollary 2.1. *Let* $(M, \mathbb{F})$ *martingale. Then for all* $c > 0$

$$c\mathbb{P}(M_n^* \geq c) \leq \mathbb{E}(|M_n| I_{\{M_n^* \geq c\}}) \leq \mathbb{E}|M_n|. \tag{2.7}$$

2.2.2. *Doobs* L^p*- inequality.* Denote $||X||_q = (\mathbb{E}|X|^q)^{\frac{1}{q}}$, when $q > 1$. Recall Hölder's inequality.

Theorem 2.5 (Hölder)**.** *Let* p, q *be conjugate numbers :* $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *If* X, Y *are random variables , which satisfy* $X \in L^p(\mathbb{P}), Y \in L^q(\mathbb{P})$*, then* $XY \in \mathcal{L}^1$ *and*

$$\mathbb{E}|XY| \leq ||X||_p ||Y||_q. \tag{2.8}$$

Proof We may assume that $X, Y \geq 0$ and also that $||X||_p > 0$ [if $||X||_p = 0$, then $X = 0$ $\mathbb{P}$- a.s.. and we have the inequality (2.8)]
Define a probability measure Q by

$$Q(A) = \frac{1}{\mathbb{E}_{\mathbb{P}} X^p} \int_A X^p d\mathbb{P}$$

and a random variable U by $U = \frac{Y}{X^{p-1}} I_{\{X>0\}}$. It follows from the Jensen inequality that

$$(\mathbb{E}_Q U)^q \leq \mathbb{E}_Q U^q.$$

On the other hand

$$\begin{aligned}\mathbb{E}_Q U &= \frac{1}{\mathbb{E}X^p}\mathbb{E}_{\mathbb{P}}\left(\frac{YX^p}{X^{p-1}}I_{\{X>0\}}\right)\\ &= \frac{1}{\mathbb{E}X^p}\mathbb{E}_{\mathbb{P}}(XYI_{\{X>0\}}) = \frac{1}{\mathbb{E}_{\mathbb{P}}X^p}\mathbb{E}_{\mathbb{P}}(XY)\end{aligned}$$

and so

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(XY) &= \mathbb{E}_P X^p \mathbb{E}_Q U \le \mathbb{E}_P X^p \left(\mathbb{E}_Q U^q\right)^{\frac{1}{q}}\\ &\le \mathbb{E}_P X^p \left(\frac{1}{\mathbb{E}_{\mathbb{P}}X^p}\mathbb{E}_P\left(\left(\frac{Y}{X^{p-1}}\right)^q I_{\{X>0\}}X^p\right)\right)^{\frac{1}{q}}\\ &= ||X||_p \left(\mathbb{E}_{\mathbb{P}}(Y^q I_{\{X>0\}})\right)^{\frac{1}{q}} \le ||X||_p ||Y||_q,\end{aligned}$$

where we used the facts $-q(p-1)+p=0$ and $1-\frac{1}{q}=\frac{1}{p}$. □

Next we prove inequality:

Lemma 2.2. *Let $X, Y \ge 0$ be random variables with*

$$c\mathbb{P}(X \ge c) \le E(YI_{\{X\ge c\}}), \tag{2.9}$$

when $c > 0$. Then the following inequality holds

$$||X||_p \le q||Y||_p \tag{2.10}$$

with conjugate numbers p, q.

Proof We can assume that $\mathbb{P}(X \ge c) > 0$, for some $c > 0$, otherwise (2.10) holds without a proof, since the left hand side is $= 0$. Further we can assume that $||Y||_p < \infty$. Let us further assume that $||X||_p < \infty$.

Using the Fubini theorem

$$\begin{aligned}\mathbb{E}X^p &= \int_\Omega \left(\int_0^{X(\omega)} pc^{p-1}dc\right)\mathbb{P}(d\omega)\\ &= \int_0^\infty \left(\int_\Omega I_{\{X(\omega)\ge c\}}\mathbb{P}(d\omega)\right)pc^{p-1}dc.\end{aligned}$$

By the inequality (2.9) and Fubini theorem we obtain

$$\begin{aligned}\int_0^\infty \mathbb{P}(X\ge c)pc^{p-1}dc &\le \int_0^\infty \mathbb{E}\left(YI_{\{X\ge c\}}\right)pc^{p-2}dc\\ &= \int_\Omega Y\left(\int_0^{X(\omega)} pc^{p-2}dc\right)\mathbb{P}(d\omega) = q\mathbb{E}(YX^{p-1})\end{aligned}$$

With the help of Hölder inequality we obtain that

$$\mathbb{E}(YX^{p-1}) \le ||X^{p-1}||_q ||Y||_p = \left(\mathbb{E}X^p\right)^{\frac{1}{q}} ||Y||_p,$$

since $(p-1)q = (p-1)\frac{p}{p-1} = p$.

So we have obtained the inequality

$$\mathbb{E}X^p \le q\mathbb{E}(YX^{p-1}) \le q||Y||_p \left(\mathbb{E}X^p\right)^{\frac{1}{q}}. \tag{2.11}$$

Because $1-\frac{1}{q}=\frac{1}{p}$ we obtain from the (2.11) by dividing the left and right hand side by the term $(\mathbb{E}X^p)^{\frac{1}{q}}$ the inequality

$$||X||_p \le q||Y||_p.$$

Now we show how one can proceed without assuming that $||X||_p < \infty$: if the random variable X satisfies (2.9), then it is true for the truncated random variable $X \wedge n$ and so for all n we have

$$||X \wedge n||_p \leq q||Y||_p.$$

The claim (2.10) follows now by letting $n \to \infty$. □

Corollary 2.2. *Let $X, Y \geq 0$ be random variables, which satisfy (2.9) and $||Y||_p < \infty$. Then $X \in L^p$.*

Theorem 2.6 (Doob's L^p inequality)**.** *Let $(X, \mathbb{F}, \mathbb{P})$ be a martingale. If $X_n \in L^p$, then $X_k^* \in L^p$ for all $k \leq n$ and*

$$||X_n^*||_p \leq q||X_n||_p. \tag{2.12}$$

Proof From the corollary 2.1 we get that for the maximal process X^* we have the inequality (2.9) for all $k \leq n$, when $Y = |X_n|$. From the corollary 2.2 we get that if $X_n \in L^p$, then also $X_k^* \in L^p$ for all $k \leq n$. The inequality (2.12) will follow from the Lemma 2.2. □

The following are left to exercises:

Let (X_n, F_n), $n = 1, \dots, N$ be a supermartingale and let $c > 0$ be a constant. Then

$$\begin{aligned} c\mathbb{P}(\max_{n \leq N} X_n \geq c) &\leq \mathbb{E}X_1 - \int_{\{\max_{n \leq N} X_n < c\}} X_N d\mathbb{P} \\ &\leq \mathbb{E}X_1 + \mathbb{E}X_N^-. \end{aligned}$$

and

$$c\mathbb{P}(\min_{n \leq N} X_n \leq -c) \leq -\int_{\{\min_{n \leq N} X_n \leq -c\}} X_N d\mathbb{P} \leq \mathbb{E}X_N^-.$$

To prove these one can use stopping times $\tau = \min\{k \leq N : X_k \geq c\}$ and $\sigma = \min\{k \leq N : X_k \leq -c\}$; in addition we agree that $\min\{\emptyset\} = N$.

2.3. Martingale convergence theorem.

2.3.1. *Doob's upcrossing inequality.*

Convergence and upcrossings. Let x_n,$n \geq 1$ be a sequence of real numbers and let $a < b$ be fixed. Let us compute, how many times the sequence x_n, $n \geq 1$, will pass over the interval $[a, b]$ in such a way that the passsing will take place from below to up: upcrossing, when $k \leq N$; denote this number by $u_N^x[a, b]$. Put $u_\infty^x[a, b] = \lim_N u_N^x[a, b]$.

We can now make the following observations concerning the convergence and upcrossings:

- If $u_\infty^x[a, b] = \infty$ for some $a < b$; then the sequence $(x_n)_{n \geq 1}$ does not converge.
- If the sequence $(x)_{n \geq 1}$ converges, then $u_\infty^x[a, b] < \infty$ for all $a < b$.
- If $u_\infty^x[a, b] < \infty$ for all $a < b$, then either the sequence $(x_n)_{n \geq 1}$ converges or it goes in absolute value to infinity: $\lim x_n \to -\infty$ or $x_n \to \infty$.

2.3.2. *Doob's upcrossing inequality.* Let $(X, \mathbb{F})$ be a process. Given $a, b \in \mathbb{R}$ with $a < b$ define two sequences of stopping times τ_n and σ_n as follows. Put $\tau_0 = 0$ and then define recursively, for $j \geq 0$

$$\sigma_{j+1} = \min\{k > \tau_j : X_k \leq a\}, \quad \tau_{j+1} = \min\{k > \sigma_{j+1} : X_k \geq b\}, \tag{2.13}$$

where $\min\{\emptyset\} = \infty$. We have then

$$u_n^X[a, b] = \max\{j : \tau_j \leq n\}.$$

Theorem 2.7 (Doob's upcrossing inequality)**.** *Let $(X, \mathbb{F})$ be a submartingale, $a < b$. Then*

$$\mathbb{E}u_n^X[a, b] \leq \frac{1}{b-a}\mathbb{E}\left((X_n - a)^+\right). \tag{2.14}$$

Proof Put $Y_n = (X_n - a)^+$. Then $(Y, \mathbb{F})$ is also a submartingale. Moreover, we have that $Y_n \geq 0$ and

$$u_n^Y[0, b-a] = u_n^X[a, b].$$

So, without loosing generality it is enough to prove the inequality

$$\mathbb{E}u_n^X[0, b] \leq \frac{1}{b}\mathbb{E}X_n$$

in the case $X_n \geq 0$ for all $n \geq 0$ [othewise we can go from X to Y, where $Y_n = (X_n - a)^+$]. Define now the two sequences of stopping times σ_n and τ_n with respect to 0 and $b > 0$. Always $\tau_n \geq n$ and so we can write

$$X_n = X_0 + \sum_{i=1}^{\infty}(X_{\tau_i \wedge n} - X_{\sigma_i \wedge n}) + \sum_{i=0}^{\infty}(X_{\sigma_{i+1} \wedge n} - X_{\tau_i \wedge n}).$$

Because X is a submartingale, we will get

$$\mathbb{E}(X_{\sigma_{i+1} \wedge n} - X_{\tau_i \wedge n}) \geq 0, i \geq 0.$$

On the other hand, bu the construction of the stopping times σ_n, τ_n we have

$$\sum_{i=1}^{\infty}(X_{\tau_i \wedge n} - X_{\sigma_i \wedge n}) \geq b u_n^X[0, b].$$

Hence we have

$$\mathbb{E}X_n \geq \mathbb{E}X_0 + \mathbb{E}\sum_{i=1}^{\infty}(X_{\tau_i \wedge n} - X_{\sigma_i \wedge n}) \geq b\mathbb{E}u_n^X[0, b],$$

and this proves (2.14). □ 18.3. 2008