Mat-1.3602 Stochastic Analysis.

Extra problem sheet *Azmoodeh/Valkeila.* Problem should be solved by May 15, 2008, and delivered to Azmoodeh.

1. Let ξ_k, $k \geq 1$, be independent random variable with $\mathbb{E}\xi_k = 0$ and $\sigma_k^2 = \mathbb{E}\xi_k^2 < \infty$ for all $k \geq 1$. Put $S_n = \sum_{k=1}^n \xi_k$. If $\sum_{k=1}^\infty \sigma_k^2 < \infty$, show that then the sequence S_n, $n \geq 1$ has an integrable almost sure limit S.
2. Let $X = (X_n, F_n, \mathbb{P})_{n\geq 1}$ be a non-negative local martingale. Show that X is a supermartingale.
3. If $X = (X_t, F_t, \mathbb{P})_{t\geq 0}$ is a nonnegative supermartingale. If $\mathbb{E}X_t = \mathbb{E}X_0$ for all $t \geq 0$, show that then X is a martingale.
4. Let X, A be a non-negative continuous processes, $X, A \in \mathbb{F}$ and τ is a stopping time. Let $\sigma = \inf\{u : X_u \geq \epsilon\}$ and $\gamma = \inf\{u : A_u \geq \eta\}$ for $\epsilon, \eta > 0$. Show that
$$\{\sup_{s\leq\tau} X_s > \epsilon\} \subset \{A_\tau \geq \eta\} \cup \{\sigma \leq \tau < \gamma\}.$$
5. Prove the following variant of Lenglart inequality: Let X be a non-negative continuous process and let A be a continuous increasing process, both adapted to $\mathbb{F}$. We say that the process A L-dominates the process X, if for any bounded stopping time σ we have that
$$\mathbb{E}X_\sigma \leq EA_\sigma.$$
Show that if A L-dominates X, then for any stopping time τ and $\eta, \epsilon > 0$
$$\mathbb{P}(X_\tau^* > \epsilon) \leq \frac{\eta}{\epsilon} + \mathbb{P}(A_\tau > \eta).$$
[Hint: use the previous exercise.]
6. Let M^n be a sequence of $(\mathbb{F}, \mathbb{P})$- local martingales and τ is a stopping time. Prove the following fact:
$$\sup_{s\leq\tau} |M_s^n| \xrightarrow{\mathbb{P}} 0 \Leftrightarrow \langle M^n, M^n\rangle_\tau \xrightarrow{\mathbb{P}} 0,$$
as $n \longrightarrow \infty$.
7. Let W be a Brownian motion. Compute $\mathbb{E}W_T^4$ with Itô formula.
8. Let W be a Brownian motion. What is the process $\langle W^2, W^2\rangle$?
9. Let $f \in C_{1,2}(\mathbb{R}_+, \mathbb{R})$ such that
$$\frac{\partial f}{\partial t} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}.$$
Let $W = (W_t)_{t\geq 0}$ be a Brownian motion and put $X_t = f(t, W_t)$. Show that the process $(X, \mathbb{F}^W)$ is a local martingale. and if in addition
$$\mathbb{E}[\int_0^T \left(\frac{\partial f}{\partial x}(t, W_t)\right)^2 dt < \infty,$$
then $X = (X_t)_{0\leq t\leq T}$ is a martingale.
10. Let W be a Brownian motion with respect to $\mathbb{F}, \mathbb{P}$). Find a measure Q such that the process $X_t = W_t + t - t^3$ is a Brownian motion on the interval $[0, T]$, $T > 0$.