Mat-1.3602 Stochastic Analysis.

Exercise 3.4. 2008 Azmoodeh/Valkeila.

1. We say that a martingale $M_n, n \geq 1$ is <u>closed from the right</u>, if there exists an integrable random variable Y such that $M_n = \mathbb{E}[Y|F_n]$.
   - Show that a martingale is closed from the right if and only if it is uniformly integrable.
   - Let $M_n, n \geq 1$ be closed from the right: $M_n \mathbb{E}[Y|F_n]$, where Y is an integrable random variable. Give an example, where $M_\infty \neq Y$.

Let (M_n, F_n) $n \geq 1$ be an uniformly integrable martingale and τ is a stopping time. Then

$$M_\tau = \mathbb{E}[M_\infty | F_\tau]. \tag{1}$$

Let $X \geq 0$ be a random variable with $\mathbb{E}_{\mathbb{P}} X = 1$. Define Q by putting

$$Q(A) = \int_A X d\mathbb{P}.$$

Show that Q is a probability measure and that $Q \prec\prec \mathbb{P}$

Let $Q \prec\prec \mathbb{P}$: then for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mathbb{P}(F) < \delta \Rightarrow Q(F) < \epsilon$.

Assume that $Q \prec\prec \mathbb{P}$ and put $Y = \frac{dQ}{d\mathbb{P}}$. Show that Y is unique $\mathbb{P}$ - almost surely.

Assume that $Q \sim \mathbb{P}$ and put $Y = \frac{dQ}{d\mathbb{P}}$. Show that $\frac{d\mathbb{P}}{dQ} = \frac{1}{Y}$.