Mat-1.3602 Stochastic Analysis.

Exercise 29.4. 2008 *Azmoodeh/Valkeila.*

1. Prove the following: a continuous square integrable centered process process X with $X_0 = 0$ is a Brownian motion if and only if $e^{aX_t - \frac{1}{2}a^2 t}$ is a martingale for $a \in \mathbb{R}$.

   [In the proof you can use the following fact: the Laplace transformation $\vartheta_X(\lambda)$ of a random variable X is defined by $\vartheta_X(\lambda) = \mathbb{E}\left(e^{\lambda X}\right)$ and we assume that the integral is well-defined. One can show that the Laplace transformation determines the law of the random variable.]
2. Let W be a Brownian motion and put $S_t = e^{\sigma W_t + \mu t - \frac{1}{2}\mu^2 t}$. Find a measure Q such that S is a martingale with respect to Q.
3. Let M, N be local martingales with $N_0 = M_0 = 0$. Show that the following formula, <u>Yor's formula</u>, is true:
   $$\mathcal{E}(M)\mathcal{E}(N) = \mathcal{E}(M + N + \langle M, N\rangle),$$
   where
   $$\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M,M\rangle_t}.$$
4. Let X be a positive martingale, and define the <u>stochastic logarithm</u> of X by the formula $(\mathcal{L}(X))_t = \int_0^t \frac{dX_s}{X_s}$ Show that if M is a continuous local martingale, then
   $$\mathcal{L}(\mathcal{E}(M)) = M$$
   and for a positive martingale Z
   $$\mathcal{E}(\mathcal{L}(Z)) = Z.$$
5. Let $f = f(t, x)$ be a continuous function with $f \in C_{1,2}$. Moreover, assume that
   $$f_t(t, x) + \frac{1}{2} f_{xx}(t, x) = 0;$$
   show that in this case the process $f(t, W_t)$ is a local martingale. Here W is a standard Brownian motion.
6. Visit OpinionsOnLine and fill in the data for this course.