Mat-1.3602 Stochastic Analysis.

Exercise 24.4. 2008 *Azmoodeh/Valkeila.*

1. Let H be a predictable process such that $\int_0^\infty H_s^2 d\langle M, M\rangle_s < \infty$, where M is a continuous martingale. Let τ be a stopping time such that $\int_0^\tau H_s^2 d\langle M, M\rangle_s \in L^1(\mathbb{P})$. Show that the following processes are the same, and they are all martingales:
$$\int_0^{\tau\wedge t} H_s dM_s, \quad \int_0^t H_s 1_{(0,\tau]}(s) dM_s \quad \text{and} \quad \int_0^t H_s dM_s^\tau.$$
2. Compute $\mathbb{E}W_t^4$ using the fact that $\mathbb{E}W_s^2 = s$ and the Itô formula.
3. Let X, Y be continuous semimartingales. Show that
$$XY = X_0 Y_0 + X \circ Y + Y \circ X + \langle X, Y\rangle.$$
4. Let h^i, where $i = 1, 2$ be a deterministic functions with $\mathbb{E}\int_0^\infty (h_s^i)^2 ds < \infty$ and W is a Brownian motion. Define $M_t = \int_0^t h_s^1 dW_s$ and $N_t = \int_0^t h_s^1 \int_0^s h_u^2 dW_u dW_s$. Show that M and N are martingales and that $\mathbb{E}(N_t M_t) = 0$.
5. Show that the process $W_t - \int_0^t \frac{W_s}{s} ds$ is a Brownian motion with respect to $\mathbb{F}^X$, but not with respect to $\mathbb{F}^W$. Hint: direct computation. Use also the fact that orthogonality is equivalent to independence in the Gaussian case, which we have here.
6. Find the value of the stochastic integral $\int_0^t W_s e^{W_s} dW_s$.