Mat-1.3602 Stochastic Analysis.

Exercise 17.4. 2008 Azmoodeh/Valkeila.

1. Let W be a standard Brownian motion and put $Y_t = e^{aW_t - \frac{1}{2}a^2 t}$, where $a \in \mathbb{R}$ is a constant. Show that Y is a martingale with $\mathbb{E}Y_t = 1$.
2. Let $(M, \mathbb{F}, \mathbb{P})$ be a continuous martingale and τ is a stopping time. Show that for $s < t$ we have
$$\mathbb{E}_{\mathbb{P}}[M_{\tau \wedge t} I_{\{\tau > s\}} | F_s] = M_s I_{\{\tau > s\}}.$$
3. Assume that M is a continuous square integrable martingale and $\langle M \rangle$ is a continuous increasing process such that the process $M^2 - \langle M \rangle$ is a martingale. Let $H = \alpha 1_{(s,t]}(\cdot)$, where $\alpha \in F_s$ and $|\alpha| \le 1$. Then the process $N_t = \int_0^t H_s dM_s$ is a martingale. What is the process $\langle N, N \rangle$? If $K = \beta 1_{(u,v]}$, where $\beta \in F_u$ and $|\beta| \le 1$, then show that $\int_0^t K_s dN_s = \int_0^t K_s H_s dM_s$.
4. [Continuation of Problem 2] Put $L_t = \int_0^t K_s dN_s$. Prove that
$$\mathbb{E}\left(\int_0^\infty K_s dN_s\right)^2 = \mathbb{E}\int_0^\infty (K_s H_s)^2 \, d\langle M, M \rangle_s.$$
What is the process $\langle L, L \rangle$?
5. Let M be a continuous square integrable martingale and $\langle M \rangle$ is a continuous increasing process such that the process $M^2 - \langle M \rangle$ is a martingale. Let $H = \alpha 1_{(s,t]}(\cdot)$, where $\alpha \in F_s$ and $|\alpha| \le 1$ and $K = \beta 1_{(u,v]}$, where $\beta \in F_u$ and $|\beta| \le 1$. Let $N = H \circ M$ and $L = K \circ M$. Compute the process $\langle N, L \rangle$. Compute the expectation $\mathbb{E}\left(\int_0^\infty H_s dM_s \int_0^\infty K_s dM_s\right)$ in terms of H, K and $\langle M, M \rangle$.
6. Let H^n be a sequence of simple predictable processes, $|H^n| \le 1$ and $(H^n)^*_\infty \to 0$ in probability, M is a continuous martingale such that $\mathbb{E}\langle M, M \rangle_\infty < \infty$. Show that $\int_0^\infty H^n_s dM_s \to$ in $L^2(\mathbb{P})$.