Mat-1.3602 Stochastic Analysis.

Exercise 27.3. 2008 Azmoodeh/Valkeila.

1.+ 2. Prove Theorem 2.1.:

Let $\mathbb{F}$ be a history, the process X satisfies $X \in \mathbb{F}$ and C is a predictable process.

- If in addition $0 \leq C_n(\omega) \leq K$ and X is a supermartingale, then $Y \doteq (C \circ X)$ is a supermartingale.
- If in addition $|C_n(\omega)| \leq K$ and X is a martingale, then $Y \doteq (C \circ X)$ is a martingale.

3. let $(M, \mathbb{F}, \mathbb{P})$ be a martingale, adn τ is a stopping time such that $\mathbb{P}(\tau < \infty) = 1$. Prove the following corollary of Theorem 2.6:

$$||X_\tau||_p \leq ||X^*_\tau||_p \leq q||X_\tau||_p,$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

4. Let $(X, \mathbb{P})$ be a submartingale. Put $Y_n(X_n - a)^+$, where $a \in \mathbb{R}$. SHow that $(Y, \mathbb{F})$ is also a submartingale.

5. + 6. Let (X_n, F_n), $n = 1, \ldots, N$ be a supermartingale and let $c > 0$ be a constant. Then

$$\begin{aligned} c\mathbb{P}(\max_{n \leq N} X_n \geq c) &\leq \mathbb{E}X_1 - \int_{\{\max_{n \leq N} X_n < c\}} X_N d\mathbb{P} \\ &\leq \mathbb{E}X_1 + \mathbb{E}X_N^-. \end{aligned}$$

and

$$c\mathbb{P}(\min_{n \leq N} X_n \leq -c) \leq -\int_{\{\min_{n \leq N} X_n \leq -c\}} X_N d\mathbb{P} \leq \mathbb{E}X_N^-.$$

To prove these one can use stopping times $\tau = \min\{k \leq N : X_k \geq c\}$ and $\sigma = \min\{k \leq N : X_k \leq -c\}$; in addition we agree that $\min\{\emptyset\} = N$.