Mat-1.3602 Stochastic Analysis.

Exercise 10.4. 2008 Azmoodeh/Valkeila.

1. X has independent increments and $X_t \in L^2(\mathbb{P})$. Then $M_t = X_t - \mathbb{E}X_t$ is a martingale and $M_t^2 - \mathrm{Var}(X_t)$ is a martingale with respect to $\mathbb{F}^X$.
2. W is a Brownian motion. What is the $L^2(\mathbb{P})$ limit of the sum $\sum_{t_k \in \pi_n} W_{t_k}(W_{t_k} - W_{t_{k-1}})$?
3. A random time τ is an optional time, if $\{\tau < t\} \in F_t$ for all $t \geq 0$. Show that τ is an optional time if and only if τ is a $\mathbb{F}_+$ stopping time. [This is another formulation of Lemma 3.1. from the handouts. Here $\mathbb{F}_+ = (F_{t+})_{t \geq 0}$.]
4. Let X be a continuous process and B is a closed set on real line. Define τ by $\tau = \inf\{t : X_t \in B\}$. Show that τ is a stopping time.
5. Let X be an $\mathbb{F}$- adapted integrable process. Show that if for every bounded stopping time $\mathbb{E}X_\tau = \mathbb{E}X_0$, then X is a martingale.
6. Let X be a non-negative supermartingale with respect to $\mathbb{F}$. Let $\tau = \inf\{t : X_t = 0\}$. If σ is a stopping time such that $\tau < \sigma$, then $X_\sigma = 0$ a.s.