Lecture 29.4. 2008

5.3.2. *Novikov's condition.* In order to know weather the candidate $\mathcal{E}(a \circ W)$ is a true martingale or not is a difficult problem. One sufficient condition for this is Novikov's condition.

But before the formulation of this theorem recall that if $\int_0^T a_s^2 ds < \infty$ and $a$ is a predictable process with respect to $(\mathbb{F}, \mathbb{P})$, then the stochastic integral $a \circ W$ is a local martingale. Then also $\mathcal{E}(a \circ W)$ is a local martingale. Because $\mathcal{E}(a \circ W)_0 = 1$ and $\mathcal{E}(a \circ W)_t > 0$, then $\mathcal{E}(a \circ W)$ is a supermartingale. We know from the extra exercise set that in order the exponential to be a true martingale on the interval $[0, T]$ it is enough to show that $\mathbb{E}_{\mathbb{P}}(\mathcal{E}(a \circ W)_T) = 1$. One sufficient condition is given in the following theorem.

**Theorem 5.6** (Novikov)**.** *Let $a$ be a predictable process such that*

$$\mathbb{P}(\int_0^t a_s^2 ds < \infty) = 1$$

*for all $0 \leq t \leq T$. Let $M = \mathcal{E}(a \circ W)$ be the exponential of $a \circ W$. If the Novikovs condition holds*

$$\mathbb{E}_{\mathbb{P}} \left( \exp \left( \frac{1}{2} \int_0^T a_s^2 ds \right) \right) < \infty,$$

*then $M_t$, $0 \leq t \leq T$ is a martingale.*

We will not prove this theorem.

5.4. **Itô-Clark representation theorem.** We know that if $W$ is a Brownian motion and $H$ is a predictable process with $\mathbb{E} \int_0^T H_d^2 s < \infty$, then the stochastic integral $H \circ W$ is a martingale and $Y := \int_0^T H_s dW_s$ is a square integrable $F_T^W$- measurable random variable. The Itô-Clark representation theorem tells that the opposite fact is true: if $Y \in L^2(F_T^W, \mathbb{P})$ then there exists a unique predictable process $H^Y$ such that

$$Y = \mathbb{E}Y + \int_0^T H_s^Y dW_s. \tag{5.10}$$

From (5.10) it follows that every square integrable $(\mathbb{F}^W, \mathbb{P})$- martingale $M$ has unique representation

$$M_t = \mathbb{E}M_T + \int_0^T H_s^M dW_s;$$

this is easy, since the square integrable $F_T^W$- measurable random variable $Y = M_T$ has the representation (5.10), and from this we obtain the representation for the martingale $M$. Before we prove (5.10) we have the following fact from (5.10):

- Every square integrable $(\mathbb{F}^W, \mathbb{P})$ martingale is continuous.

**Theorem 5.7.** *Let $W$ be a Brownian motion, and $Y$ is a square integrable $F_T^W$- measurable random variable. Then $Y$ has a unique integral representation with a predictable process $H^Y$ and Brownian motion $W$:*

$$Y = \mathbb{E}Y + \int_0^T H_s^Y dW_s. \tag{5.11}$$

*Proof* Note first that the representation (5.11) is unique: If we have another representation with predictable $\tilde{H}$, then

$$0 = Y - Y = \int_0^T \left(\tilde{H}_s - H_s^Y\right) dW_s$$

and the Itô- isometry gives

$$\mathbb{E} \int_0^T \left(\tilde{H}_s - H_s^Y\right)^2 ds = 0,$$

and hence $H^Y$ is unique [in the space $L^2(\mathcal{P}(\mathbb{F}^W), \mathbb{P} \otimes Leb)$].
Before the proof, we recall the following fact: if $Y \in L^2(F_T^W)$ then there exists a sequence of continuous bounded functions on $\mathbb{R}^{k_n}$ such that $f^n(W_{t_1^n}, \dots, W_{t_{k_n}^n}$ such that $f^n \to Y$ in $L^2(\mathbb{P})$.
Next we assume that $Y = f(W_{t_1}, W_{t_2}, \dots, W_{t_n})$ is a square integrable random variable and $f$ is smooth; here $0 \le t_1 < t_2 < \cdots t_n \le T$.
Define a function $U$ by

$$U(t_1, x_1, t_2, x_2, \dots, t_{n-1}, x_{n-1}, t, x)$$
$$= \int_{\mathbb{R}} f(x_1, \dots, x_{n-1}, y) \frac{1}{\sqrt{2\pi(t_n - t)}} \exp\left[-\frac{(y-x)^2}{2(t_n - t)}\right] dy.$$

Note first that

$$f(W_{t_1}, W_{t_2}, \dots, W_{t_n}) = U(t_1, W_{t_1}, \dots, t_n, W_{t_n}).$$

The function $U$ satisfies

$$\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial x^2} = 0;$$

Itô- formula gives now

$$U(t_1, x_1, t_2, x_2, \dots, t_{n-1}, x_{n-1}, t_n, W_{t_n}) - V(t_1, x_1, t_2, x_2, \dots, t_{n-1}, x_{n-1})$$
$$= \int_{t_{n-1}}^{t_n} U_x(t_1, x_{t_1}, t_2, x_{t_2}, \dots, t_{n-1}, x_{{}_{n-1}}, s, W_s) dW_s,$$

where $V(t_1, x_1, t_2, x_2, \dots, t_{n-1}, x_{n-1}) = U(t_1, x_1, t_2, \dots, t_{n-1}, x_{n-1}, t_{n-1}, x_{n-1})$.
We can now continue backwards from $n-1$ to $n-2$, and in this way we obtain the representation (5.10).
The general claim is proved as follows: every $F_T^W$ measurable random variable $Y$ is a limit of smooth $n$-dimensional functionals of $W$, where $n \to \infty$..
Denote by $H^m$ the predictable processes, which are associated to representation of $f_m$, the smooth approximation of $Y$. Then $H^m$ is a c-sequence, which has a limit $H^Y$; by Itô isometry we get the representation (5.10). □

5.5. **Stochastic differential equations.** In this section we shall work with a fixed Brownian motion $W$; moreover we have two functions $\sigma$ and $b$, both are functions from $\mathbb{R}_+ \prod \mathbb{R}$ to $\mathbb{R}$. We want to specify, when we can write equations of the form

$$X_t(\omega) = \zeta(\omega) + \int_s^t \sigma(u, X_u) dW_s + \int_s^t b(u, X_u) du, \tag{5.12}$$

where $\zeta(\omega)$ is a $F_s$- measurable random variable.

We shall assume that the coefficients $\sigma$ and $b$ satisfy the following two conditions:

- Lipschitz continuity:
$$|\sigma(t,x)-\sigma(t,y)| \leq A|x-y|$$
and
$$|b(t,x)-b(t,y)| \leq A|x-y|.$$
- linear growth:
$$|\sigma(t,x)| \leq A(1+|x|)$$
and $|b(t,x)| \leq A(1+|x|)$

Note that linear growth and Lipschitz continuity imply that $|\sigma(t,0)| \leq A$ and $|b(t,0)| \leq A$.
Assume that $\zeta \in F_s$ is square integrable, we look for a progressively measurable continuous solution $x$ to (5.12) with

$$\mathbb{E}_{\mathbb{P}}\left\{\int_s^t x_u^2(\omega)du\right\} < \infty. \tag{5.13}$$

**Theorem 5.8.** *Assume that the coefficients $b,\sigma$ satisfy linear growth and Lipschitz continuity assumptions, $\zeta \in L^2(F_s)$. Then there is a unique solution to (5.12) with the property (5.13).*

The proof is based on Lindelöf-Picard iteration:
*Proof* **(i)** Define $x_t^0$ by $x_t^0 \equiv \zeta$, and then recursively

$$x_t^{n+1} = \zeta + \int_s^t \sigma(u,x_u^n)dW_u + \int_s^t b(u,x_u^n)du.$$

We have the following facts, which can be checked by induction:

- For $n \geq 0$, $x^{n+1}$ is well defined, progressively measurable and almost surely continuous.
- By linear growth we obtain that for every $t > s$
$$\sup_{s\leq u\leq t} \mathbb{E}_{\mathbb{P}}\left\{|x_u^{n+1}|^2\right\} < \infty,$$
and this together with linear growth implies

$$\mathbb{E}_{\mathbb{P}}\left\{\int_s^t |\sigma(u,x_u^{n+1})|^2du\right\} < \infty. \tag{5.14}$$

The property (5.14) means that the stochastic integral in the next recursion step is well defined.
**(ii)** Define, for $t \geq s$ the difference

$$\begin{aligned} y_t^n &= x_t^{n+1} - x_t^n \\ &= \int_s^t \left[\sigma(u,x_u^n) - \sigma(u,x_u^{n-1})\right] dW_u \\ &\quad + \int_s^t \left[b(u,x_u^n) - b(u,x_u^{n-1})\right] du \end{aligned}$$

Denote $\Delta_n(t) = \mathbb{E}_{\mathbb{P}}\left\{(y_t^n)^2\right\}$ we obtain using Lipschitz continuity, Itô- isometry and Schwartz inequality that

$$\Delta_n(t) \le 2A^2(1+t-s)\int_s^t \Delta_{n-1}(u)du.$$

Fix now $T \ge t \ge s$ and we can write the above as

$$\Delta_n(t) \le 2A^2(1+T-s)\int_s^t \Delta_{n-1}(u)du = C_T\int_s^t \Delta_{n-1}(u)du. \tag{5.15}$$

If $n = 0$ we define

$$y_t^0 = \int_0^t \sigma(u,\zeta)dW_u + \int_0^t b(u,\zeta)du$$

and estimate directly

$$\Delta_0(t) = \mathbb{E}_{\mathbb{P}}\left\{\left(y_t^0\right)^2\right\} \le CC_Tt, \tag{5.16}$$

where $C_T$ is as in (5.15) and $C = \mathbb{E}_{\mathbb{P}}\left\{1+\zeta^2\right\}$.
**(iii)** Iterating the inequalities (5.15) and (5.16) give the estimate

$$\Delta_n(t) \le C\frac{C_T^{n+1}t^{n+1}}{(n+1)!} \tag{5.17}$$

and so $\sup_{s\le t\le T}\Delta_n(t) \to 0$. Next, consider the expression $||\sup_{s\le t\le T}|y_t^n|||_{L^2(\mathbb{P})}$. We have that

$$||\sup_{s\le t\le T}|y_t^n|||_{L^2(\mathbb{P})} \le ||\sup_{s\le t\le T}|y_n^{(1)}(t)|||_{L^2(\mathbb{P})} + ||\sup_{s\le t\le T}|y_n^{(2)}(t)|||_{L^2(\mathbb{P})}.$$

with

$$y_n^{(1)}(t) = \int_s^t \left[\sigma(u,x_u^n) - \sigma(u,x_u^{n-1}\right] dW_u$$

and

$$y_n^{(2)}(t) = \int_s^t \left[b(u,x_u^n) - b(u,x_u^{n-1})\right] du.$$

The process $y_n^{(1)}$ is a martingale, and so with the help of the Doob maximal inequality we get

$$||\sup_{s\le t\le T}|y_n^{(1)}(t)|||_{L^2(\mathbb{P})} \le 2||y_n^{(1)}(T)||_{L^2(\mathbb{P})} \le A_T\left\{\int_s^T \Delta_{n-1}(u)du\right\}^{\frac{1}{2}}.$$

For the second term we will use the trick $\left(\int_s^t f_u du\right)^2 \le (t-s)\int_s^t f_u^2 du$ and we obtain

$$||\sup_s \le t \le T|y_n^{(2)}(t)|||_{L^2(\mathbb{P})} \le A_T\left\{\int_s^T \Delta_{n-1}(u)du\right\}^{\frac{1}{2}}.$$

**(iv)** Putting together the estimates from above we obtain that

$$\sum_n ||\sup_{t\le T}|y_n(t)|||_{L^2(\mathbb{P})} < \infty.$$

Hence the sum $\sum y^n$ uniformly and almost surely on every finite interval. This means that $x^n$ converges uniformly almost surely to a limit $X$, which is continuous and progressively measurable. We can write the following display:

$$x_t^{n+1} = \zeta + \int_s^t \sigma(u, x_u^n) dW_u + \int_s^t b(u, x_u) du;$$

here $x^{n+1} \to X$ uniformly almost surely and $\int_s^t b(u, x_u^n)du \to \int_s^t b(u, X_u)du$ almost surely, and so $X$ is a solution to (5.12).

**(v)** Finally, we show the uniqueness. If $\tilde{X}$ is another solution to (5.12) with the property (5.13), then by the previous estimates we get for $Y = X - \tilde{X}$ the estimate

$$\mathbb{E}_{\mathbb{P}} \left\{ Y_t^2 \right\} \leq C_T \int_s^t \mathbb{E}_{\mathbb{P}} \left( Y_u^2 \right) du.$$

On the other hand we also have the estimate

$$\mathbb{E}_{\mathbb{P}} Y_t^2 \leq C_T t,$$

valid for $s \leq t \leq T$. Iterating this together with the previous one gives

$$\mathbb{E}_{\mathbb{P}} Y_t^2 \leq \frac{C_T^n t^n}{n!}$$

and this show that $Y_t = 0$ for all $t \leq T$. □

That's all, folks.