Helsinki University of Technology Somersalo/Bingham
Mathematics

Mat-1.198 Scattering Theory

7th set of exercises, 26.3.2003

1. Let $A(k,\widehat{x},\alpha)$, $k>0$ being the wave number, be the scattering amplitude for potential scattering,
$$A(k,\widehat{x},\alpha)=\int_D e^{-ik\widehat{x}\cdot y}q(y)u(y,k,\alpha)dy, \tag{1}$$
where $u(y,k,\alpha)$ is the solution of the Lippmann-Schwinger equation,
$$u(x,k,\alpha)=e^{ik\alpha\cdot x}-\int_D \Phi(x-y,k)q(y)u(y,k,\alpha)dy.$$
We can extend the wave number to negative values, too, by substituting $-k$ to the Lippmann-Schwinger equation. Here, we assume that q is real valued and if it depends on k as in the acoustic scattering, it is an even function of k. The corresponding solution satisfies no longer the outgoing radiation condition. However, it allows us to extend A to negative wave numbers, too, by the formula (1) above.

   Prove that for this extension, we have
$$A(-k,\widehat{x},\alpha)=\overline{A(k,\widehat{x},\alpha)}=A(-k,-\alpha,-\widehat{x}).$$
You may assume that $n=3$.

2. The Born approximation for the scattering amplitude in material scattering is
$$A(k,\widehat{x},\alpha)\approx A_{\rm B}(k,\widehat{x},\alpha)=\int_D e^{-ik(\widehat{x}-\alpha)\cdot y}q(y)dy,$$
i.e., $A_{\rm B}$ is the Fourier transform of q evaluated at
$$\xi=k(\widehat{x}-\alpha)\in\mathbb{R}^n.$$
Assume that the potential q is independent of the wave number k. (This assumption is valid in quantum mechanical scattering.) Then q itself can be reconstructed approximately by applying the inverse Fourier transform on A. The problem is how to integrate A that is a function defined on $\mathbb{R}_+\times S^{n-1}\times S^{n-1}$.

   Prove the following integration identities in the case $n=3$:
$$\begin{aligned}\int_{\mathbb{R}^3} f(\xi)d\xi &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{S^2} f(k(\widehat{x}-\alpha))|\widehat{x}-\alpha|^2 d\alpha k^2 dk\\ &= \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{S^2}\int_{S^2} f(k(\widehat{x}-\alpha))|\widehat{x}-\alpha|^2 d\alpha d\widehat{x} k^2 dk.\end{aligned}$$

Hence, we obtain an approximation for q as

$$q(x) \approx q_{\mathrm{B}}(x) = \left(\frac{1}{2\pi}\right)^3 \frac{1}{8\pi} \int_{-\infty}^{\infty} \int_{S^2} \int_{S^2} e^{ik(\widehat{x}-\alpha)\cdot x} A(k, \widehat{x}, \alpha) |\widehat{x} - \alpha|^2 d\alpha d\widehat{x} k^2 dk.$$

(Hint: Let $\widehat{x}$ be a fixed parameter. If $\xi = k(\widehat{x} - \alpha)$, show that

$$k = \frac{|\xi|}{2\widehat{x} \cdot \widehat{\xi}}, \quad \alpha = \widehat{x} - 2(\widehat{x} \cdot \widehat{\xi})\widehat{\xi},$$

where $\widehat{\xi} = \xi/|\xi|$. These formulas can be used to calculate the Jacobian.)

3. What are the corresponding formulas of the previous problem in $\mathbb{R}^2$?