Helsinki University of Technology Somersalo/Bingham
Mathematics

Mat-1.198 Scattering Theory

6th set of exercises, 19.3.2003

1. Let $u(x,\alpha)$, $x \in \mathbb{R}^3 \setminus \overline{D}$ be the solution of the scattering problem by a sound-soft obstacle D,
$$u(x,\alpha) = e^{ik\alpha\cdot x} + u_{\rm sc}(x,\alpha),$$
and $A(\widehat{x},\alpha)$ the corresponding far field pattern. Prove the reciprocity relation
$$A(\widehat{x},\alpha) = A(-\alpha,-\widehat{x}).$$
Hint: Starting from the Helmholtz representation for $u_{\rm sc}$, show that
$$A(\widehat{x},\alpha) = \frac{1}{4\pi}\int_{\partial D}\left(u_{\rm sc}(y,\alpha)\frac{\partial u}{\partial n}(y,-\widehat{x}) - u(y,-\widehat{x})\frac{\partial u_{\rm sc}}{\partial n}(y,\alpha)\right)dS$$
and similarly,
$$A(-\alpha,-\widehat{x}) = \frac{1}{4\pi}\int_{\partial D}\left(u(y,-\widehat{x})\frac{\partial u_{\rm inc}}{\partial n}(y,\alpha) - u_{\rm inc}(y,\alpha)\frac{\partial u}{\partial n}(y,-\widehat{x})\right)dS.$$
Deduce the result from the above representations.

2. Prove the above reciprocity relation for the far field pattern of the solution of the Lippmann-Schwinger equation,
$$u(x,\alpha) = e^{ik\alpha\cdot x} - \int_D \Phi(x-y)q(y)u(y,\alpha)dy.$$
Hint: Use the same formulas as above, integration being over the surface of a sphere $|x| = R$.

3. Derive the following two-potential formula: Let $A_j(\widehat{x},\alpha)$, $j = 1,2$ be the far field patterns corresponding to the compactly supported potentials q_j, $j = 1,2$, respectively, and denote by $u_j(x,\alpha)$ the corresponding acoustic fields with incoming plane wave with the direction α. Show that
$$A_1(\widehat{x},\alpha) - A_2(\widehat{x},\alpha) = \frac{1}{4\pi}\int u_1(y,\alpha)(q_1(y)-q_2(y))u_2(y,-\widehat{x})dy.$$
This formula is central in inverse scattering theory.
Hint: Write
$$\begin{aligned} A_1(\widehat{x},\alpha) &= \frac{1}{4\pi}\int e^{-ik\widehat{x}\cdot y}q_1(y)u_1(y,\alpha)dy \\ &= \frac{1}{4\pi}\int \big(u_2(y,-\widehat{x}) - u_{2,\rm sc}(y,-\widehat{x})\big)q_1(y)u_1(y,\alpha)dy, \end{aligned}$$
and a similar formula for $A_2(\widehat{x},\alpha) = A_2(-\alpha,-\widehat{x})$ and subtract.