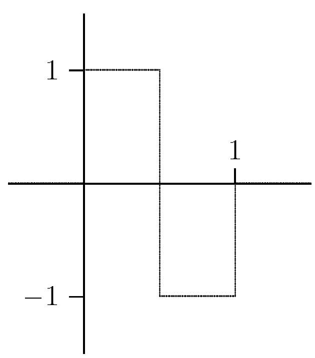
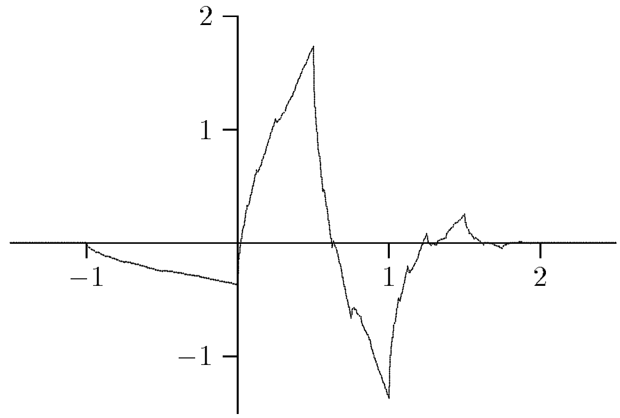
Introduction

1. Basic ideas

By wavelets one usually means a family of functions generated from one single function ψ by the operation of dilations and translations, that is, a family of the form $\{|a|^{-1/2}\psi((\bullet - b)/a)\}$ where a and b are real numbers. Another view of the subject is to restrict oneself to a function ψ (called the mother wavelet) with the property that the set $\{2^{-\frac{j}{2}}\psi(2^{-j}\bullet - k\}_{k,j\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R};\mathbb{C})$. The simplest example of such a function ψ is provided by the Haar function



that is studied in greater detail in Section 2. It is quite easy to see that this function is a wavelet in the sense described above, but it is far from obvious that one can find smoother functions or functions having other desirable properties. For example the following function is a wavelet



but it is not immediately obvious.

2. The Haar system

Define the function φ

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in [0,1), \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that the functions $\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$ spans the subspace of $L^2(\mathbb{R})$ that consists of all square integrable functions functions that are constant on the intervals of the form $[k, k+1)$. Denote this subspace by V_0. Moreover, it is easy to see that $\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$ is in fact an orthonormal basis for V_0. Let V_{-1} be the subspace of all square integrable functions that are constant on intervals of the form $[2^{-1}k, 2^{-1}(k+1))$. An orthonormal basis for this space is given by $\{2^{1/2}\varphi(2\bullet - k)\}_{k\in\mathbb{Z}}$. Obviously $V_0 \subset V_{-1}$ and therefore it must be possible to express the function φ in terms of the functions $\varphi(2\bullet - k)$ and we have $\varphi = 2\sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\bullet - k)$ where $\alpha(0) = \alpha(1) = \frac{1}{2}$ and $\alpha(k) = 0$ if $k \neq 0, 1$. (Here it may seem to be stupid to have the numer 2 in front of the sum, but it turns out that if this 2 is not put there, then it turns up in places where it is more of a nuisance.)

Denote by W_0 the orthogonal complement of V_0 in V_{-1}. It is not difficult to see that an orthonormal basis for W_0 is given by $\{\psi(\bullet - k)\}_{k\in\mathbb{Z}}$ where

$$\psi(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}), \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that we have $\psi = 2\sum_{k\in\mathbb{Z}}(-1)^k\alpha(1-k)\varphi(2\bullet - k) = \varphi(2\bullet) - \varphi(2\bullet - 1)$.

This argument can be extended to show that $\{2^{-\frac{m}{2}}\varphi(2^{-m}\bullet -k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_m, the space of all square integrable functions that are constant on intervals of the form $[2^m k, 2^m(k+1))$, and that $\{2^{-\frac{m}{2}}\psi(2^{-m}\bullet -k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for the orthogonal complement of V_m in V_{m-1}. Thus we see that if $m < j$ we have $V_m = V_j \oplus W_j \oplus W_{j-1} \oplus \ldots \oplus W_{m+1}$ and also that $\bigcup_{m=-\infty}^{j} V_m$ is dense in $L^2(\mathbb{R};\mathbb{C})$. We say that the Haar function φ generates a multiresolution of $L^2(\mathbb{R};\mathbb{C})$ and that ψ is the mother wavelet while φ is the scaling function or father wavelet.

The main weakness of the Haar functions is that they are not even continuous, and therefore there are good reasons to try to generalize the ideas above to other cases as well, even if the simplicity of the Haar functions is lost.