TKK, Institute of mathematics Gripenberg
Mat-1.3422 Wavelet theory
Exercise 5
10.10-17.10.2006

1. Let $N_1(\underline{t}) = \chi_{[0,1)}(\underline{t})$, that is $N_1(t) = 1$ when $0 \le t < 1$ and $N_1(t) = 0$ otherwise and let $N_m = N_{m-1} * N_1$ when $m \ge 2$. Show that one can find a sequnce c such that $(\varphi(\underline{x}-n))_{n\in Z}$ is an orthonormal sequnce in the space $L^2(\mathbb{R})$ when $\varphi(\underline{x}) = \sum_{k\in\mathbb{Z}} c(k)N_2(x-k)$. (You may assume that $\sum_{k\in\mathbb{Z}} |\widehat{N_2}(\omega+k)|^2 = \frac{1}{3} + \frac{2}{3}\cos(\pi\omega)^2$.) Is it true that $\varphi(\underline{x}) = \sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\underline{x}-k)$ for some sequnce α (using the fact that $N_2(\underline{x}) = 2\sum_{k=0}^{2} a(k)N_2(2\underline{x}-k)$ where $a(0) = a(2) = \frac{1}{4}$ and $a(1) = \frac{1}{2}$)?

2. Let

$$\Phi(\underline{t}) = \int_{\mathbb{R}} \varphi(x)\varphi(x+\underline{t})\,\mathrm{d}x,$$

where φ is the function constructed in the previous exercise. Show that Φ is a twice continuously differentiable function which on each interval $(n, n+1)$ is a polynomial of degree at most 3, that is, Φ is the fundamental interpolation function for cubic interpolation.

3. Let $N > 1$ and $P_N(\underline{x}) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} \underline{x}^k$ so that we have $(1-\underline{x})^N P_N(\underline{x}) + \underline{x}^N P_N(1-\underline{x}) = 1$. In addition, let α_N be such that $\hat{\alpha}_N(\underline{\omega}) = (\frac{1}{2}(1+\mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}}))^N Q_N(\mathrm{e}^{-\mathrm{i}2\pi\Omega})$ where $|Q_N(\mathrm{e}^{-\mathrm{i}2\pi\Omega})|^2 = P_N(\sin(\pi\omega)^2)$.

(a) Calculate $P_N(\frac{1}{2})$.
(b) Show that $y^{-N+1}P_N(y) > x^{-N+1}P_N(x)$ kun $0 < y < x$.
(c) Show that $P_N(x) < x^{N-1}2^{2N-2}$ kun $\frac{1}{2} < x < 1$.
(d) Show that

$$\lim_{N\to\infty} |\hat{\alpha}_N(\omega)| = \begin{cases} 1, & |\omega| < \frac{1}{4}, \\ 0, & \frac{1}{4} < |\omega| < \frac{3}{4}. \end{cases}$$

4. Let

$$w(t) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k \underline{t}),$$

where $0 < a < 1$ and $b > 1$. In addition, let

$$\psi(t) = \begin{cases} \mathrm{e}^{-(t-\frac{1}{b})^{-2}-(t-b)^{-2}+(1-\frac{1}{b})^{-2}+(1-b)^{-2}}, & \frac{1}{b} < t < b, \\ 0, & t \le \frac{1}{b} \text{ or } t \ge b, \end{cases}$$

so that we know that ψ and $\hat{\psi} \in \mathcal{C}_{\downarrow}^{\infty}(\mathbb{R})$. Define

$$f_j(\underline{t}) = b^j \int_{-\infty}^{\infty} w(\underline{t}-s)\hat{\psi}(b^j s)\,\mathrm{d}s.$$

Show that

$$f_j(\underline{t}) = \tfrac{1}{2}a^j \mathrm{e}^{-\mathrm{i}2\pi b^j \underline{t}}.$$

5. Let w, ψ and f_j, $j \geq 1$ be as in the previous exercise. Assume that w is differentiable in the point t, and define the function v by the formula

$$v(s) = \begin{cases} \dfrac{w(t-s) - w(t) + sw'(t)}{s}, & s \neq 0, \\ 0, & s = 0, \end{cases}$$

Is v continuous and bounded? Show, by writing $w(t - \underline{s}) = \underline{s}v(\underline{s}) + w(t) - \underline{s}w'(t)$, that

$$f_j(t) = b^{-j} \int_{-\infty}^{\infty} sv\left(b^{-j}s\right) \hat{\psi}(s)\, \mathrm{d}s.$$

Show with the aid of this result that $\lim_{j\to\infty} b^j f_j(t) = 0$. For which values of the product ab does this contradict the result of the previous exercise?