TKK, Institute of mathematics — Gripenberg
Mat-1.3422 Wavelet theory
Exercise 4
3.10-11.10.2006

1. Assume that the function f is m times continuously differentiable, supp f is compact but not empty (i.e., $f \not\equiv 0$), and that

$$f(\underline{x}) = 2\sum_{k=a_-}^{a_+} \alpha(k) f(2\underline{x}-k).$$

Show that none of the sequences $(f^{(j)}(k))_{k\in\mathbb{Z}}$, $0 \le j \le m$, is the zero-sequence.

2. Determine the numbers c_{-2}, c_{-1}, c_1, and c_2 so that if p is a polynomial of degree 3 for which $p(x_0 - \frac{3}{2}h) = f_{-2}$, $p(x_0 - \frac{1}{2}h) = f_{-1}$, $p(x_0 + \frac{1}{2}h) = f_1$, and $p(x_0 + \frac{3}{2}h) = f_2$ then $p(x_0) = c_{-2}f_{-2} + c_{-1}f_{-1} + c_1 f_1 + c_2 f_2$.
Hint: Write $p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)$ *so that you only have to find* a_0.

3. Let α be the following sequence:

$$\begin{aligned}
\alpha(0) &= \tfrac{1}{8}(1+\sqrt{3}),\\
\alpha(1) &= \tfrac{1}{8}(3+\sqrt{3}),\\
\alpha(2) &= \tfrac{1}{8}(3-\sqrt{3}),\\
\alpha(3) &= \tfrac{1}{8}(1-\sqrt{3}),\\
\alpha(k) &= 0 \quad \text{muuten.}
\end{aligned}$$

Calculate the sequence $\gamma(n) = \sum_{j\in\mathbb{Z}} \alpha(j)\alpha(j+n)$ when n is odd. (According to earlier calculations we know that $\gamma(2n) = \frac{1}{2}\delta_{0,n}$.)

How are these numbers related to the numbers in the previous exercis?

4. Let α be a sequence such that ä $\alpha(0) = \alpha(2) = \frac{1}{2}$ and $\alpha(n) = 0$ otherwise. Define the functions F_j, $j \ge 0$ so that $F_0(n) = \delta_{0,n}$,

$$F_{j+1}(2^{-j-1}n) = 2\sum_{k\in\mathbb{Z}} \alpha(n-2k)F_j(2^{-j}k), \quad n\in\mathbb{Z}, \quad j\ge 0,$$

and for all other values of the argument the function F_j are determined by linear interpolation, that is, $F_j(\underline{x}) = \sum_{n\in\mathbb{Z}} F_j(2^{-j}n)w(2^j\underline{x}-n)$ where $w(\underline{x}) = \max\{0, 1-|\underline{x}|\}$. What happens to the functions F_j when $j\to\infty$.

5. Assume that the following claim holds: If $\psi \in L^2(\mathbb{R})$ then $(2^{-\frac{m}{2}}\psi(2^{-m} \bullet - k))_{m,k\in\mathbb{Z}}$ is an orthonormal basis in the space $L^2(\mathbb{R})$ if and only if

$$\sum_{m\in\mathbb{Z}} \left|\hat{\psi}(2^m \bullet)\right|^2 \overset{\text{a.e.}}{=} 1,$$

and

$$\sum_{p=0}^{\infty} \hat{\psi}(2^p \bullet)\overline{\hat{\psi}\big(2^p(\bullet + k)\big)} \overset{\text{a.e.}}{=} 0 \quad \text{for all odd integers } k.$$

(a) Is $(2^{-\frac{m}{2}}\psi(2^{-m} \bullet - k))_{m,k\in\mathbb{Z}}$ an orthonormal basis in the space $L^2(\mathbb{R})$ if $\hat{\psi}(\omega) = 1$ when $\frac{1}{2} \le |\omega| \le 1$ and 0 otherwise?

(b) If now $\psi \in L^2(\mathbb{R})$ is such that $\left(2^{-\frac{m}{2}}\psi(2^{-m} \bullet - k)\right)_{m,k\in\mathbb{Z}}$ is an orthonormal basus in the space $L^2(\mathbb{R})$, and if ϕ is the Hilbert transform of on ψ, that is $\phi(\underline{t}) = \lim_{\epsilon\downarrow 0} \frac{1}{\pi} \int_{|\underline{t}-s|\ge\epsilon} \frac{\psi(s)}{\underline{t}-s}\,\mathrm{d}s$, is then $(2^{-\frac{m}{2}}\phi(2^{-m} \bullet - k))_{m,k\in\mathbb{Z}}$ an orthonormal basis in $L^2(\mathbb{R})$ as well?

Hint: $\hat{\phi}(\underline{\omega}) = -\mathrm{i}\,\mathrm{sign}\,(\underline{\omega})\hat{\psi}(\underline{\omega})$.