TKK, Institute of mathematics Gripenberg
Mat-1.3422 Wavelet theory
Exercise 1
12-19.9.2006

1. Show that the sequence $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis for the Hilbert space H (i.e., $\langle e_n, e_m\rangle = \delta_{n,m}$ and $\overline{\mathrm{span}\{e_n\}_{n\in\mathbb{Z}}} = H$) if and only if

$$\|e_n\| = 1, \quad n \in \mathbb{Z} \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |\langle f, e_n\rangle|^2 = \|f\|^2, \quad f \in H.$$

2. Let $\phi \in L^2(\mathbb{R};\mathbb{C})$. Show that the following conditions are equivalent (for some constants A and B):

(a)

$$A\sum_{k\in\mathbb{Z}}|c_k|^2 \leq \left\|\sum_{k\in\mathbb{Z}} c_k\phi(\bullet - k)\right\|^2_{L^2(\mathbb{R})} \leq B\sum_{k\in\mathbb{Z}}|c_k|^2,$$

for each sequence $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$;

(b)

$$A \overset{\text{a.e.}}{\leq} \sum_{m\in\mathbb{Z}}\left|\hat{\phi}(\bullet + m)\right|^2 \overset{\text{a.e.}}{\leq} B$$

Hint: Let $f = \sum_{k\in\mathbb{Z}} c_k\phi(\bullet - k)$ and calculate the Fourier transform $\hat{f}$. Show in addition that $\int_{\mathbb{R}}|\hat{f}(\omega)|^2\,\mathrm{d}\omega = \int_0^1\sum_{m\in\mathbb{Z}}|\hat{f}(\omega+m)|^2\,\mathrm{d}\omega$, and recall that the Fourier transform of c is periodic with period 1. Observe also that if $E \subset [0,1]$ is measurable with positive measure (e.g. a nonempty interval), then one can find a sequence $c \in \ell^2(\mathbb{Z})$ so that $\hat{c}(\omega) \overset{\text{a.e.}}{=} 1$ when $\omega \in E$ and 0 otherwise.

3. Show that if φ is such that $(\varphi(\bullet - k))_{k\in\mathbb{Z}}$ is an orthonoprmal set, then $\lim_{m\to\infty} P_m f \to 0$ (in $L^2(\mathbb{R})$) for every $f \in L^2(\mathbb{R})$ where P_m is the orthogonal projection onto the closed subspace V_m spanned by the functions $(2^{-\frac{m}{2}}\varphi(2^{-m}\bullet - k))_{k\in\mathbb{Z}}$.
Hint: You may assume as known that $(2^{-\frac{m}{2}}\varphi(2^{-m}\bullet - k))_{k\in\mathbb{Z}}$ is an orthonormal set and that if $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal set in some hilbert space H and P is the projection onto the closed subspace space spanned by this set, then $\|Pu\|^2 = \sum_{n\in\mathbb{Z}}|\langle u, e_n\rangle|^2$. Why is it sufficient to take as function f the characteristic function of an interval?

4. Let $\alpha(-1) = \alpha(2) = \frac{1}{2}$ and $\alpha(k) = 0$ when $k \neq -1, 2$. Calculate the Fourier transform $\hat{\alpha}(\underline{\omega}) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi\underline{\omega}k}\alpha(k)$ of the sequence α and check that

$$\left|\hat{\alpha}(\omega)\right|^2 + \left|\hat{\alpha}(\omega + \tfrac{1}{2})\right|^2 = 1,$$

for all ω.

5. Let $\alpha(-1) = \alpha(2) = \frac{1}{2}$ and $\alpha(k) = 0$ when $k \neq -1, 2$. Determine the function φ so that

$$\varphi(x) = 2\sum_{k\in\mathbb{Z}}\alpha(k)\varphi(2x - k), \quad x \in \mathbb{R},$$

and so that $\int_{\mathbb{R}}\varphi(x)\,\mathrm{d}x = 1$. Is $(\varphi(\bullet - k))_{k\in\mathbb{Z}}$ and orthonormal set in $L^2(\mathbb{R})$?
Hint: The function φ is not very complicated and can be found by trial and error.