7 New topologies from old ones

In this section families of mappings transfer (induce and co-induce) topologies from topological spaces to a set in natural ways. The most important cases for us are quotient and product spaces.

Comparison of topologies. If (X, τ_1) and (X, τ_2) are topological spaces and $\tau_1 \subset \tau_2$, we say that τ_1 is *weaker* than τ_2 and τ_2 is *stronger* than τ_1.

7.1 Co-induction

Co-induced topology. Let X and J be sets, (X_j, τ_j) be topological spaces for every $j \in J$, and $\mathcal{F} = \{f_j : X_j \to X \mid j \in J\}$ be a family mappings. The *$\mathcal{F}$-co-induced topology* of X is the strongest topology τ on X such that the mappings f_j are continuous for every $j \in J$. Indeed, this definition is sound, because

$$\tau = \{U \subset X \mid \forall j \in J : \ f_j^{-1}(U) \in \tau_j\},$$

as the reader may easily verify.

Example. Let $\mathcal{A}$ be a topological vector space and $\mathcal{J}$ its subspace. Let us denote $[x] := x + \mathcal{J}$ for $x \in \mathcal{A}$. Then the quotient topology of $\mathcal{A}/\mathcal{J} = \{[x] \mid x \in \mathcal{A}\}$ is the $\{(x \mapsto [x]) : \mathcal{A} \to \mathcal{A}/\mathcal{J}\}$-co-induced topology.

Example. Let (X, τ_X) be a topological space. Let $R \subset X \times X$ be an equivalence relation. Let

$$[x] := \{y \in X \mid (x, y) \in R\},$$

$$X/R := \{[x] \mid x \in X\},$$

and define the *quotient map* $p : X \to X/R$ by $x \mapsto [x]$. The *quotient topology* of the *quotient space* X/R is the $\{p\}$-co-induced topology on X/R. Notice that X/R is compact if X is compact, since $p : X \to X/R$ is a continuous surjection.

Remark. The message of the following exercise is that if our compact space X is not Hausdorff, we "factor out" inessential information that $C(X)$ "does not see" to obtain a compact Hausdorff space related nicely to X.

Exercise*. Let X be a topological space, and define $C \subset X \times X$ by

$$(x, y) \in C \stackrel{\text{definition}}{\iff} \forall f \in C(X): \ f(x) = f(y).$$

Prove:
(a) C is an equivalence relation on X.
(b) There is a natural bijection between the sets $C(X)$ and $C(X/C)$.
(c) X/C is a Hausdorff space.
(d) If X is a compact Hausdorff space then $X \cong X/C$.

Exercise. For $A \subset X$ the notation X/A means X/R_A, where the equivalence relation R_A is given by

$$(x, y) \in R_A \stackrel{\text{definition}}{\iff} x = y \text{ or } \{x, y\} \subset A.$$

Let X be a topological space, and let $\infty \subset X$ be a closed subset. Prove that the mapping

$$X \setminus \infty \to (X/\infty) \setminus \{\infty\}, \quad x \mapsto [x],$$

is a homeomorphism.

Finally, let us state a basic property of co-induced topologies:

Proposition. *Let X have the $\mathcal{F}$-co-induced topology, and Y be a topological space. A mapping $g : X \to Y$ is continuous if and only if $g \circ f$ is continuous for every $f \in \mathcal{F}$.*

Proof. If g is continuous then the composed mapping $g \circ f$ is continuous for every $f \in \mathcal{F}$.

Conversely, suppose $g \circ f_j$ is continuous for every $f_j \in \mathcal{F}$, $f_j : X_j \to X$. Let $V \subset Y$ be open. Then

$$f_j^{-1}(g^{-1}(V)) = (g \circ f_j)^{-1}(V) \subset X_j \quad \text{is open;}$$

thereby $g^{-1}(V) = f_j(f_j^{-1}(g^{-1}(V))) \subset X$ is open □

Corollary. *Let X, Y be topological spaces, R be an equivalence relation on X, and endow X/R with the quotient topology. A mapping $f : X/R \to Y$ is continuous if and only if $(x \mapsto f([x])) : X \to Y$ is continuous* □

7.2 Induction

Induced topology. Let X and J be sets, (X_j, τ_j) be topological spaces for every $j \in J$ and $\mathcal{F} = \{f_j : X \to X_j \mid j \in J\}$ be a family of mappings. The *$\mathcal{F}$-induced topology* of X is the weakest topology τ on X such that the mappings f_j are continuous for every $j \in J$.

Example. Let (X, τ_X) be a topological space, $A \subset X$, and let $\iota : A \to X$ be defined by $\iota(a) = a$. Then the $\{\iota\}$-induced topology on A is

$$\tau_X|_A := \{U \cap A \mid U \in \tau_X\}.$$

This is called the *relative topology* of A. Let $f : X \to Y$. The restriction $f|_A = f \circ \iota : A \to Y$ satisfies $f|_A(a) = f(a)$ for every $a \in A \subset X$.

Exercise. Prove **Tietze's Extension Theorem**: *Let X be a compact Hausdorff space, $K \subset X$ closed and $f \in C(K)$. Then there exists $F \in C(X)$ such that $F|_K = f$.*

Example. Let (X, τ) be a topological space. Let σ be the $C(X) = C(X, \tau)$-induced topology, i.e. the weakest topology on X making the all τ-continuous functions continuous. Obviously, $\sigma \subset \tau$, and $C(X, \sigma) = C(X, \tau)$. If (X, τ) is a compact Hausdorff space it is easy to check that $\sigma = \tau$.

Example. Let X, Y be topological spaces with bases $\mathcal{B}_X, \mathcal{B}_Y$, respectively. Recall that the product topology for $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ has a base

$$\{U \times V \mid U \in \mathcal{B}_X,\ V \in \mathcal{B}_Y\}.$$

This topology is actually induced by the family

$$\{p_X : X \times Y \to X,\ p_Y : X \times Y \to Y\},$$

where the *coordinate projections* p_X and p_Y are defined by $p_X((x, y)) = x$ and $p_Y((x, y)) = y$.

Product topology. Let X_j be a set for every $j \in J$. The *Cartesian product*

$$X = \prod_{j \in J} X_j$$

is the set of the mappings

$$x : J \to \bigcup_{j \in J} X_j \quad \text{such that} \quad \forall j \in J : \ x(j) \in X_j.$$

Due to the Axiom of Choice, X is non-empty if all X_j are non-empty. The mapping

$$p_j : X \to X_j, \quad x \mapsto x_j := x(j),$$

is called the jth *coordinate projection*. Let (X_j, τ_j) be topological spaces. Let $X := \prod_{j \in J} X_j$ be the Cartesian product. Then the $\{p_j \mid j \in J\}$-induced topology on X is called the *product topology* of X.

If $X_j = Y$ for all $j \in J$, it is customary to write

$$\prod_{j \in J} X_j = Y^J = \{f \mid f : J \to Y\}.$$

Weak*-topology. Let $x \mapsto \|x\|$ be the norm of a normed vector space X over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The dual space $X' = \mathcal{L}(X, \mathbb{K})$ of X is set of bounded linear functionals $f : X \to \mathbb{K}$, having a norm

$$\|f\| := \sup_{x \in X:\ \|x\| \leq 1} |f(x)|.$$

This endows X' with a Banach space structure. However, it is often better to use a weaker topology for the dual: Let us define $x(f) := f(x)$ for every $x \in X$ and $f \in X'$; this gives the interpretation $X \subset X'' := \mathcal{L}(X', \mathbb{K})$, because

$$|x(f)| = |f(x)| \leq \|f\| \ \|x\|.$$

So we may treat X as a set of functions $X' \to \mathbb{K}$, and we define the *weak*-topology* of X' to be the X-induced topology of X'.

Let us state a basic property of induced topologies:

Proposition. *Let X have the $\mathcal{F}$-induced topology, and Y be a topological space. A mapping $g : Y \to X$ is continuous if and only if $f \circ g$ is continuous for every $f \in \mathcal{F}$.*

Proof. If g is continuous then the composed mapping $f \circ g$ is continuous for every $f \in \mathcal{F}$.

Conversely, suppose $f_j \circ g$ is continuous for every $f_j \in \mathcal{F}$, $f : X \to X_j$. Let $y \in Y$, $V \subset X$ be open, $g(y) \in V$. Then there exist $\{f_{j_k}\}_{k=1}^n \subset \mathcal{F}$ and open sets $W_{j_k} \subset X_{j_k}$ such that such that

$$g(y) \in \bigcap_{k=1}^{n} f_{j_k}^{-1}(W_{j_k}) \subset V.$$

Let

$$U := \bigcap_{k=1}^{n} (f_{j_k} \circ g)^{-1}(W_{j_k}).$$

Then $U \subset Y$ is open, $y \in U$, and $g(U) \subset V$; hence $g : Y \to X$ is continuous at an arbitrary point $y \in Y$, i.e. $g \in C(Y, X)$ $\square$

Hausdorff preserved in products: It is easy to see that a Cartesian product of Hausdorff spaces is always Hausdorff: If $X = \prod_{j \in J} X_j$ and $x, y \in X$, $x \neq y$, then there exists $j \in J$ such that $x_j \neq y_j$. Therefore there are open sets $U_j, V_j \subset X_j$ such that

$$x_j \in U_j, \quad y_j \in V_j, \quad U_j \cap V_j = \emptyset.$$

Let $U := p_j^{-1}(U_j)$ and $V := p^{-1}(V_j)$. Then $U, V \subset X$ are open,

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

Also compactness is preserved in products; this is stated in Tihonov's Theorem (Tychonoff's Theorem). Before proving this we introduce a tool:

Non-Empty Finite InterSection (NEFIS) property. Let X be a set. Let $NEFIS(X)$ be the set of those families $\mathcal{F} \subset \mathcal{P}(X)$ such that every finite subfamily of $\mathcal{F}$ has a non-empty intersection. In other words, a family $\mathcal{F} \subset \mathcal{P}(X)$ belongs to $NEFIS(X)$ if and only if $\bigcap \mathcal{F}' \neq \emptyset$ for every finite subfamily $\mathcal{F}' \subset \mathcal{F}$.

Lemma. *A topological space X is compact if and only if $\mathcal{F} \notin NEFIS(X)$ whenever $\mathcal{F} \subset \mathcal{P}(X)$ is a family of closed sets satisfying $\bigcap \mathcal{F} = \emptyset$.*

Proof. Let X be a **set**, $\mathcal{U} \subset \mathcal{P}(X)$, and $\mathcal{F} := \{X \setminus U \mid U \in \mathcal{U}\}$. Then

$$\bigcap \mathcal{F} = \bigcap_{U \in \mathcal{U}} (X \setminus U) = X \setminus \bigcup \mathcal{U},$$

so that $\mathcal{U}$ is a cover of X if and only if $\bigcap \mathcal{F} = \emptyset$. Now the claim follows the definition of compactness $\square$

Tihonov's Theorem (1935). *Let X_j be a compact space for every $j \in J$. Then $X = \prod_{j \in J} X_j$ is compact.*

Proof. To avoid the trivial case, suppose $X_j \neq \emptyset$ for every $j \in J$. Let $\mathcal{F} \in NEFIS(X)$ be a family of closed sets. In order to prove the compactness of X we have to show that $\bigcap \mathcal{F} \neq \emptyset$.

Let

$$P := \{\mathcal{G} \in NEFIS(X) \mid \mathcal{F} \subset \mathcal{G}\}.$$

Let us equip the set P with a partial order relation $\leq$:

$$\mathcal{G} \leq \mathcal{H} \overset{\text{definition}}{\iff} \mathcal{G} \subset \mathcal{H}.$$

The **Hausdorff Maximal Principle** says that the chain $\{\mathcal{F}\} \subset \mathcal{P}$ belongs to a maximal chain $C \subset P$. The reader may verify that $\mathcal{G} := \bigcup C \in P$ is a maximal element of P.

Notice that the maximal element $\mathcal{G}$ may contain non-closed sets. For every $j \in J$ the family

$$\{p_j(G) \mid G \in \mathcal{G}\}$$

belongs to $NEFIS(X_j)$. Define

$$\mathcal{G}_j := \{\overline{p_j(G)} \mid G \in \mathcal{G}\}.$$

Clearly also $\mathcal{G}_j \in NEFIS(X_j)$, and the elements of $\mathcal{G}_j$ are closed sets in X_j. Since X_j is compact, $\bigcap \mathcal{G}_j \neq \emptyset$. Hence we may choose

$$x_j \in \bigcap \mathcal{G}_j.$$

The **Axiom of Choice** provides the existence of the element $x := (x_j)_{j \in J} \in X$. We shall show that $x \in \bigcap \mathcal{F}$, which proves Tihonov's Theorem.

If $V_j \subset X_j$ is a neighborhood of x_j and $G \in \mathcal{G}$ then

$$p_j(G) \cap V_j \neq \emptyset,$$

because $x_j \in \overline{p_j(G)}$. Thus

$$G \cap p_j^{-1}(V_j) \neq \emptyset$$

for every $G \in \mathcal{G}$, so that $\mathcal{G} \cup \{p_j^{-1}(V_j)\}$ belongs to P; the maximality of $\mathcal{G}$ implies that

$$p_j^{-1}(V_j) \in \mathcal{G}.$$

Let $V \in \tau_X$ be a neighborhood of x. Due to the definition of the product topology,

$$x \in \bigcap_{k=1}^{n} p_{j_k}^{-1}(V_{j_k}) \subset V$$

for some finite index set $\{j_k\}_{k=1}^{n} \subset J$, where $V_{j_k} \subset X_{j_k}$ is a neighborhood of x_{j_k}. Due to the maximality of $\mathcal{G}$, any finite intersection of members of $\mathcal{G}$ belongs to $\mathcal{G}$, so that

$$\bigcap_{k=1}^{n} p_{j_k}^{-1}(V_{j_k}) \in \mathcal{G}.$$

Therefore for every $G \in \mathcal{G}$ and $V \in \mathcal{V}_{\tau_X}(x)$ we have

$$G \cap V \neq \emptyset.$$

Hence $x \in \overline{G}$ for every $G \in \mathcal{G}$, yielding

$$x \in \bigcap_{G \in \mathcal{G}} \overline{G} \overset{\mathcal{F} \subset \mathcal{G}}{\subset} \bigcap_{F \in \mathcal{F}} \overline{F} = \bigcap_{F \in \mathcal{F}} F = \bigcap \mathcal{F},$$

so that $\bigcap \mathcal{F} \neq \emptyset$ □

Remark. Actually, Tihonov's Theorem is equivalent to the Axiom of Choice; we shall not prove this.

Banach–Alaoglu Theorem (1940). *Let X be a normed $\mathbb{C}$-vector space (or a normed $\mathbb{R}$-vector space). The norm-closed unit ball*

$$K := \overline{B_{X'}(0,1)} = \{\phi \in X' : \ \|\phi\|_{X'} \leq 1\}$$

of the dual space X' is weak-compact.*

Proof. Due to Tihonov,

$$P := \prod_{x \in X} \{\lambda \in \mathbb{C} : \ |\lambda| \leq \|x\|\} = \overline{\mathbb{D}(0, \|x\|)}^{X}$$

is compact in the product topology τ_P. Any element $f \in P$ is a mapping

$$f : X \to \mathbb{C} \quad \text{such that} \quad f(x) \leq \|x\|.$$

Hence $K = X' \cap P$. Let τ_1 and τ_2 be the relative topologies of K inherited from the weak*-topology $\tau_{X'}$ of X' and the product topology τ_P of P, respectively. We shall prove that $\tau_1 = \tau_2$ and that $K \subset P$ is closed; this would show that K is a compact Hausdorff space.

First, let $\phi \in X'$, $f \in P$, $S \subset X$, and $\delta > 0$. Define

$$\begin{aligned} U(\phi, S, \delta) &:= \{\psi \in X' : \ x \in S \Rightarrow |\psi x - \phi x| < \delta\}, \\ V(f, S, \delta) &:= \{g \in P : \ x \in S \Rightarrow |g(x) - f(x)| < \delta\}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{U} &:= \{U(\phi, S, \delta) \mid \phi \in X', \ S \subset X \text{ finite}, \ \delta > 0\}, \\ \mathcal{V} &:= \{V(f, S, \delta) \mid f \in P, \ S \subset X \text{ finite}, \ \delta > 0\} \end{aligned}$$

are bases for the topologies $\tau_{X'}$ and τ_P, respectively. Clearly

$$K \cap U(\phi, S, \delta) = K \cap V(\phi, S, \delta),$$

so that the topologies $\tau_{X'}$ and τ_P agree on K, i.e. $\tau_1 = \tau_2$.

Still we have to show that $K \subset P$ is closed. Let $f \in \overline{K} \subset P$. First we show that f is linear. Take $x, y \in X$, $\lambda, \mu \in \mathbb{C}$ and $\delta > 0$. Choose $\phi_\delta \in K$ such that

$$f \in V(\phi_\delta, \{x, y, \lambda x + \mu y\}, \delta).$$

Then

$$\begin{aligned} & |f(\lambda x + \mu y) - (\lambda f(x) + \mu f(y))| \\ \leq \ & |f(\lambda x + \mu y) - \phi_\delta(\lambda x + \mu y)| + |\phi_\delta(\lambda x + \mu y) - (\lambda f(x) + \mu f(y))| \\ = \ & |f(\lambda x + \mu y) - \phi_\delta(\lambda x + \mu y)| + |\lambda(\phi_\delta x - f(x)) + \mu(\phi_\delta y - f(y))| \\ \leq \ & |f(\lambda x + \mu y) - \phi_\delta(\lambda x + \mu y)| + |\lambda| \ |\phi_\delta x - f(x)| + |\mu| \ |\phi_\delta y - f(y)| \\ \leq \ & \delta \ (1 + |\lambda| + |\mu|). \end{aligned}$$

This holds for every $\delta > 0$, so that actually

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y),$$

f is linear! Moreover, $\|f\| \leq 1$, because

$$|f(x)| \leq |f(x) - \phi_\delta x| + |\phi_\delta x| \leq \delta + \|x\|.$$

Hence $f \in K$, K is closed □

Remark. The Banach–Alaoglu Theorem implies that a bounded weak*-closed subset of the dual space is a compact Hausdorff space in the relative weak*-topology. However, in a normed space norm-closed balls are compact if and only if the dimension is finite!