Appendix on complex analysis

Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is called *holomorphic in* Ω, denoted by $f \in H(\Omega)$, if the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for every $z \in \Omega$. Then Cauchy's integral formula provides a power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

converging uniformly on the compact subsets of the disk

$$\mathbb{D}(a,r) = \{z \in \mathbb{C} : \ |z-a| < r\} \subset \Omega;$$

here $c_n = f^{(n)}(a)/n!$, where $f^{(0)} = f$ and $f^{(n+1)} = f^{(n)\prime}$.

Liouville's Theorem. *Let $f \in H(\mathbb{C})$ such that $|f|$ is bounded. Then f is constant, i.e. $f(z) \equiv f(0)$ for every $z \in \mathbb{C}$.*

Proof. Since $f \in H(\mathbb{C})$, we have a power series representation

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

converging uniformly on the compact sets in the complex plane. Thereby

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\phi})|^2 \ \mathrm{d}\phi &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n,m} c_n \ \overline{c_m} \ r^{n+m} \ e^{i(n-m)\phi} \ \mathrm{d}\phi \\ &= \sum_{n,m} c_n \ \overline{c_m} \ r^{n+m} \ \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\phi} \ \mathrm{d}\phi \\ &= \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \end{aligned}$$

for every $r > 0$. Hence the fact

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\phi})|^2 \ \mathrm{d}\phi \le \sup_{z \in \mathbb{C}} |f(z)|^2 < \infty$$

implies $c_n = 0$ for every $n \ge 1$; thus $f(z) \equiv c_0 = f(0)$ □