Appendix on Axiom of Choice

It may be surprising, but the Zermelo-Fraenkel axiom system does not imply the following statement (nor its negation):

Axiom of Choice for Cartesian Products: *The Cartesian product of non-empty sets is non-empty.*

Nowadays there are hundreds of equivalent formulations for the Axiom of Choice. Next we present other famous variants: the classical Axiom of Choice, the Law of Trichotomy, the Well-Ordering Axiom, the Hausdorff Maximal Principle and Zorn's Lemma. Their equivalence is shown in [12].

Axiom of Choice: *For every non-empty set J there is a function $f : \mathcal{P}(J) \to J$ such that $f(I) \in I$ when $I \neq \emptyset$.*

Let A, B be sets. We write $A \sim B$ if there exists a bijection $f : A \to B$, and $A \leq B$ if there is a set $C \subset B$ such that $A \sim C$. Notion $A < B$ means $A \leq B$ such that not $A \sim B$.

Law of Trichotomy: *Let A, B be sets. Then $A < B$, $A \sim B$ or $B < A$.*

A set X is *partially ordered* with an *order relation* $R \subset X \times X$ if R is reflexive ($(x, x) \in R$), antisymmetric ($(x, y), (y, x) \in R \Rightarrow x = y$) and transitive ($(x, y), (y, z) \in R \Rightarrow (x, z) \in R$). A subset $C \subset X$ is a *chain* if $(x, y) \in R$ or $(y, x) \in R$ for every $x, y \in C$. An element $x \in X$ is *maximal* if $(x, y) \in R$ implies $y = x$.

Well-Ordering Axiom: *Every set is a chain for some order relation.*

Hausdorff Maximal Principle: *Any chain is contained in a maximal chain.*

Zorn's Lemma: *A non-empty partially ordered set where every chain has an upper bound has a maximal element.*