**Mat-1.3651 Numerical Linear Algebra**, spring 2008
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**Exercise 6** (28.2.2008)

These are held in the computer classroom Y339b (close to Y313). Please hand in the exercises marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise. In addition to that, hand in the exercises marked with [Comp. hand-in] in the *next* exercise session (13th March, that is).

* 1. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$. Show that there exists a Hermitian positive semidefinite $P$ and a $U \in \mathbb{C}^{m\times n}$ with orthonormal columns, such that $A = PU$. Furthermore: $P^2 = AA^*$.
Note: this is called the *polar decomposition* of $A$, due to reminiscence with that of a complex number $z = r\,e^{i\theta}$ where $r \geq 0$ and $|e^{i\theta}| = 1$.

* 2. Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 0.0001 \\ 4.0001 \end{pmatrix}.$$

(a) What are the matrices $A^+$ (pseudoinverse) and $P$ (orthogonal projection to $\mathcal{R}(A)$) ?

(b) Find the exact solutions $x$ and $y = Ax$ to the least squares problem $Ax \approx b$.

(c) What are $\kappa(A)$, $\theta$, and $\eta$ from the lectures?

3. Consider the polynomial $p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$.

(a) Plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ evaluating $p$ via its coefficients. (Input the coefficients either by hand or by using `expand`.)

(b) Produce the same plot again, this time evaluating $p$ via the expression $(x-2)^9$.

4. [Comp. hand-in] Take $m = 50$, $n = 12$. Using Matlab's `linspace`, define $t$ to be the $m$-vector corresponding to linearly spaced grid points from 0 to 1. Using Matlab's `vander` and `fliplr`, define $A$ to be the $m \times n$ matrix associated with least squares fitting on this grid by a polynomial of degree $n-1$. Take $b$ to be the function $\cos(4t)$ evaluated on the grid. Now, calculate and print (to sixteen-digit precision) the least squares coefficient vector $x$ by six methods:

(a) Formation and solution of the normal equations, using Matlab's \ ,

(b) QR factorization computed by `mgs` (modified Gram-Schmidt, Exercise 4)

(c) QR factorization computed by `house` (Householder triangularization, Exercise 4)

(d) QR factorization computed by Matlab's `qr` (also Householder triangularization),

(e) `x = A\ b` in Matlab (based on QR)

(f) SVD, using Matlab's `svd`.

(g) Compare and comment the results from (a)-(f).

5. [Comp. hand-in] Here we stick to the real matrices. A *random square matrix* is an $m \times m$ matrix whose entries are random numbers, independently sampled from the normal distribution with zero mean and standard deviation $m^{-1/2}$. Explore certain properties of random matrices. In Matlab, use `A = randn(m,m)/sqrt(m)`.

(a) The factor $m^{-1/2}$ gives "normalized" results as $m \to \infty$. Test this by looking at the matrix norms with/without the normalization, for example:

```
for m=1:100,
for j=1:10,
nor(m,j)=norm(randn(m,m));
nor2(m,j)=nor(m,j)/sqrt(m);
end,end
```

What do you observe? (Naturally, you modify the program as you find suitable.)

(b) What do the eigenvalues of a random matrix look like? What happens, if you take e.g. 100 random matrices and plot their eigenvalues in a single picture? Do this for $m = 8, 16, 32, \ldots$ and comment on the pattern. How does the spectral radius (Exercise 2) $\rho(A)$ behave as $m \to \infty$?

(c) How does the 2-norm of a random matrix behave as $m \to \infty$? We know that $\rho(A) \leq \|A\|$, does this appear to approach an equality?

(d) How about the smallest singular values $\sigma_{\min}$ (which are quite like the condition numbers)? First, fix $m$ and see what proportions of random matrices seem to have $\sigma_{\min} \leq \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. Then, how does the situation change as $m$ changes?

(e) How do the answers to (b)-(d) change if we use random triangular matrices instead of the full ones? (Matlab's `triu` might be useful.)

6. Experiments show that random triangular matrices with entries $\pm 1$ are exponentially ill-conditioned in the following sense: if $A \in \mathbb{C}^{m \times m}$ is such a matrix and $\kappa_m$ denotes its 2-norm condition number, then $\lim_{m \to \infty} (\kappa_m)^{1/m} = C$ for some constant $1 < C < 1.5$. Perform numerical experiments involving random matrices of various dimensions to estimate $C$ to at least 10% accuracy.