HUT , Institute of mathematics — Gripenberg
Mat-1.196 Mathematics of neural networks
Exercise 1
15–22.1.2002

1. Why is it reasonable to call the function $\sigma(\mathbf{w}\cdot\mathbf{x}-\tau)$ a *ridge* function? Is this the case in dimension 1 as well?

Solution: If $\mathbf{x}=s\mathbf{v}+\mathbf{u}$ where $\mathbf{v}\perp\mathbf{w}$, then $\sigma(\mathbf{w}\cdot\mathbf{x}-\tau)=\sigma(\mathbf{w}\cdot\mathbf{u}-\tau)$, that is, this function is a constant on lines perpendicular to $\mathbf{w}$, and drawing the graph of the function $\sigma(\mathbf{w}\cdot\mathbf{x}-\tau)$ in the case where the dimension is 2 one gets a picture looking like a ridge, provided, of course, σ is increasing on $(-\infty,a)$ and decreasing on (a,∞) for some a. In dimension 1 there are no directions perpendicular to $\mathbf{w}$ (except 0), so one does not really get a ridge in this case.

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2. Show that the polynomials (of one variable) of degree at most k are not dense in the spaces $\mathcal{C}([a,b])$ (continuous functions on $[a,b]$).

Solution: Suppose that this is not the case, so that the polynomials of degree at most k are dense in $\mathcal{C}([a,b])$. Let $x_j=a+j\frac{b-a}{k+1}$, $j=0,1,\ldots,k+1$. Let $f(x_j)=(-1)^j$ for $j=0,1,\ldots,k+1$ and define $f(x)$ by piecewise linear interpolation at all other points of $[a,b]$. Then $f\in\mathcal{C}([a,b])$ and if the polynomials of degree at most k are dense in $\mathcal{C}([a,b])$ then there is a polynomial P_k so that $\sup_{x\in[a,b]}|P_k(x)-f(x)|<\frac{1}{2}$. But then $(-1)^jP_k(x_j)>0$ for $j=0,1,\ldots,x_{k+1}$ and it follows from the intermediate value theorem that P_k has at least $k+1$ different zeros. But this is impossible and from this contradiction the claim follows.

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3. Show that if $d\geq 1$ and σ is a polynomial, then $S_d(\sigma)$ is not dense in $\mathcal{C}(\mathbb{R}^d)$ when $S_d(\sigma)=\operatorname{span}\{\,\mathbf{x}\in\mathbb{R}^d\mapsto\sigma(\mathbf{w}\cdot\mathbf{x}-\tau)\mid\mathbf{w}\in\mathbb{R}^d,\quad\tau\in\mathbb{R}\,\}$.

Solution: If σ is a polynomial of degree at most k, then the function $x\mapsto\sigma(\mathbf{w}\cdot(x,0,\ldots,0)-\tau)$ is also a polynomial of degree at most k. Thus we see that if $f\in S_d(\sigma)$, then the function $g(x)=f((x,0,\ldots,0))$ is a polynomial of degree at most k and since these polynomials are not dense in $\mathcal{C}(\mathbb{R})$ it follows that $S_d(\sigma)$ cannot be dense in $\mathcal{C}(\mathbb{R}^d)$.

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4. Show that

$$d(f,g)=\sum_{j=1}^{\infty}2^{-j}\frac{\|f-g\|_{\mathcal{B}^\infty(B_j(0)}}{1+\|f-g\|_{\mathcal{B}^\infty(B_j(0))}},$$

is a metric in the space $\mathcal{C}(\mathbb{R}^d)$ where $B_j(0)=\{\,\mathbf{x}\in\mathbb{R}^d\mid|x|<j\,\}$ and $\|f\|_{\mathcal{B}^\infty(K)}=\sup_{\mathbf{x}\in K}|f(\mathbf{x})|$.

Solution: It is clear that $0\leq d(f,g)=d(g,f)<\infty$ for all f and $g\in\mathcal{C}(\mathbb{R}^d)$ and that $d(f,g)=0$ if and only if $f=g$. Thus is remains to prove the triangle inequlity. If f, g, and $h\in\mathcal{C}(\mathbb{R}^d)$ and $j\geq 1$, then we clearly have

$$\|f-g\|_{\mathcal{B}^\infty(B_j(0)}\leq\|f-h\|_{\mathcal{B}^\infty(B_j(0)}+\|h-g\|_{\mathcal{B}^\infty(B_j(0)}.$$

Thus it suffices to show that if $a \leq b + c$ where a, b and c are nonnegative numbers, then

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

Since the function $t \mapsto \frac{t}{1+t}$ is increasing, we have have

$$\frac{a}{1+a} \leq \frac{b+c}{1+b+c},$$

and since a straightforward calculation shows that

$$\frac{b}{1+b} + \frac{c}{1+c} - \frac{b+c}{1+b+c} = \frac{(2+b+c)bc}{(1+a)(1+b)(1+b+c)} \geq 0,$$

the claim follows.