Fourier analysis (MS-C1420)

Ville Turunen

Aalto University

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Classification of signals

In this course, we divide signals into two classes:

$$\begin{aligned} (A) \quad & \textbf{Analog} \quad s:\mathbb{R}\to\mathbb{C} \quad (\text{continuous time } t\in\mathbb{R}), \\ (D) \quad & \textbf{Digital} \quad s:\mathbb{Z}\to\mathbb{C} \quad (\text{discrete time } t\in\mathbb{Z}). \end{aligned}$$

Moreover, we split these classes into two parts: a signal can be

$$\begin{aligned} \textit{either} \quad (0) \quad & \textbf{non}-\textbf{periodic} \\ \textit{or} \quad (1) \quad & \textbf{periodic}: \; s(t+p)=s(t). \end{aligned}$$

We shall study the connections between cases $(A0),(A1),(D0),(D1)$.
Fourier methods in this course:

$$\begin{aligned} \textit{Fourier integrals} \;\; (A0), \qquad & \textit{Fourier coefficients} \;\; (A1), \\ \textit{Fourier series} \;\; (D0), \qquad & \textit{DFT or FFT} \;\; (D1). \end{aligned}$$

Examples: sound, pictures, video; physical measurements; technology and sciences (1-dimensional signals in these notes).

Reminder: operations with complex numbers

Identify the point $(x, y) \in \mathbb{R} \times \mathbb{R}$ in plane
and the complex number $x + \mathrm{i}y \in \mathbb{C}$, where i is the imaginary unit.
Interpretation: real number $x \in \mathbb{R}$ is same as $x + \mathrm{i}0 \in \mathbb{C}$.

Real part	$\mathrm{Re}(x + \mathrm{i}y) := x \in \mathbb{R}$.
Imaginary part	$\mathrm{Im}(x + \mathrm{i}y) := y \in \mathbb{R}$.
Complex conjugate	$(x + \mathrm{i}y)^* = \overline{x + \mathrm{i}y} := x - \mathrm{i}y \in \mathbb{C}$.
Absolute value	$\lvert x + \mathrm{i}y \rvert := (x^2 + y^2)^{1/2} \in \mathbb{R}^+$.

Operations: e.g. $-(a + \mathrm{i}b) := -a + \mathrm{i}(-b)$ and

$$\begin{aligned} (a + \mathrm{i}b) + (x + \mathrm{i}y) &:= (a + x) + \mathrm{i}(b + y), \\ (a + \mathrm{i}b)(x + \mathrm{i}y) &:= (ax - by) + \mathrm{i}(ay + bx), \end{aligned}$$

especially $\mathrm{i}^2 = (0 + \mathrm{i}1)^2 = (0 + \mathrm{i}1)(0 + \mathrm{i}1) = -1$.
Euler's formula $\mathrm{e}^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)$,
and then $\mathrm{e}^{\mathrm{i}(\alpha+\beta)} = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta}$.

Analog non-periodic world (A0)

Continuous time ($t \in \mathbb{R}$) signal $s : \mathbb{R} \to \mathbb{C}$ has "**energy**"

$$E(s) := \|s\|^2 = \int_{\mathbb{R}} |s(t)|^2 \,\mathrm{d}t = \int_{-\infty}^{\infty} |s(t)|^2 \,\mathrm{d}t. \tag{1}$$

Denote $s \in L^2(\mathbb{R})$ if $\|s\| < \infty$.
For example, $t \in \mathbb{R}$ time (or position), and $s(t) \in \mathbb{C}$ pressure/temperature/luminosity/position/wave function...
The **Fourier (integral) transform** of signal $s : \mathbb{R} \to \mathbb{C}$ is signal $\mathcal{F}_{\mathbb{R}}(s) = \widehat{s} : \mathbb{R} \to \mathbb{C}$,

$$\widehat{s}(\nu) := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu}\, s(t) \,\mathrm{d}t = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu}\, s(t) \,\mathrm{d}t. \tag{2}$$

Variable $\nu \in \mathbb{R}$ is called "**frequency**".
Notice that $|\widehat{s}(\nu)| \leq \int_{\mathbb{R}} |s(t)| \,\mathrm{d}t.$

Example: differentiation and Fourier transform

Fourier transform changes differentiation to polynomial multiplication (and vice versa), because if $r(t) = t\,s(t)$ ja $\widehat{r} = \widehat{t\,s}$ then

$$
\begin{aligned}
\widehat{s}\,'(\nu) &= -\mathrm{i}2\pi\ \widehat{t\,s}\,(\nu), && (3)\\
\widehat{s'}\,(\nu) &= +\mathrm{i}2\pi\nu\,\widehat{s}(\nu). && (4)
\end{aligned}
$$

Let us prove the latter formula (assuming $s(t) \to 0$ as $|t| \to \infty$):

$$
\begin{aligned}
\widehat{s'}\,(\nu) &= \int_{\mathbb{R}} s'(t)\,\mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ \mathrm{d}t \\
&\overset{\textit{integrate by parts}}{=} -\int_{\mathbb{R}} s(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ \mathrm{d}t \\
&= -\int_{\mathbb{R}} s(t)\,\mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\,(-\mathrm{i}2\pi\nu)\ \mathrm{d}t \\
&= +\mathrm{i}2\pi\nu\,\widehat{s}(\nu).
\end{aligned}
$$

Example: Fourier transform of Gaussian

Gauss' normal distribution $\varphi_{\mu,\sigma}$
(mean $\mu \in \mathbb{R}$, standard deviation $\sigma > 0$):

$$\varphi_{\mu,\sigma}(t) := \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2} \left(\frac{t-\mu}{\sigma} \right)^2 \right).$$

Let $s(t) := \varphi_{0,\sigma}(t)$, so that

$$\begin{aligned}
& s'(t) = \frac{-t}{\sigma^2}\, s(t) \\
\overset{\text{previous example}}{\Longrightarrow} \quad & \mathrm{i}2\pi\nu\, \widehat{s}(\nu) = \frac{1}{\mathrm{i}2\pi\sigma^2}\, \widehat{s}\,'(\nu) \\
\Longleftrightarrow \quad & \widehat{s}\,'(\nu) = -(2\pi\sigma)^2 \nu\, \widehat{s}(\nu) \\
\Longleftrightarrow \quad & \widehat{s}(\nu) = \widehat{s}(0)\, \mathrm{e}^{-2(\pi\sigma\nu)^2}.
\end{aligned}$$

Now we must find $\widehat{s}(0)$...

... Gaussian example continues...

$$
\begin{aligned}
\widehat{s}(0) &= \int_{\mathbb{R}} s(t)\,\mathrm{d}t \\
&= \left[\int_{\mathbb{R}}\int_{\mathbb{R}} s(t)\,s(u)\,\mathrm{d}t\,\mathrm{d}u\right]^{1/2} \\
&= \left[\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi\sigma^2}\,\mathrm{e}^{-(t^2+u^2)/(2\sigma^2)}\,\mathrm{d}t\,\mathrm{d}u\right]^{1/2} \\
&\overset{\textit{polar coordinates}}{=} \left[\int_0^\infty\int_0^{2\pi} \frac{1}{2\pi\sigma^2}\,\mathrm{e}^{-r^2/(2\sigma^2)}\,r\mathrm{d}\theta\,\mathrm{d}r\right]^{1/2} \\
&= \left[\int_0^\infty \frac{r}{\sigma^2}\,\mathrm{e}^{-r^2/(2\sigma^2)}\,\mathrm{d}r\right]^{1/2} = 1,
\end{aligned}
$$

where we changed to the polar coordinates (r,θ), where $(t,u) = (r\cos(\theta), r\sin(\theta))$. Thus

$$
\widehat{\varphi_{0,\sigma}}(\nu) = \mathrm{e}^{-2(\pi\sigma\nu)^2}. \tag{5}
$$

Normal distribution and point values of signal

We calculated

$$\varphi_{0,\sigma}(t) := \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-(t/\sigma)^2/2} \quad\Longrightarrow\quad \widehat{\varphi_{0,\sigma}}(\nu) = \mathrm{e}^{-2(\pi\sigma\nu)^2},$$

so that for a "nice enough" $s : \mathbb{R} \to \mathbb{C}$, we have

$$\begin{aligned} s(t) &= \lim_{0<\sigma\to 0} \int_{\mathbb{R}} s(u)\,\varphi_{0,\sigma}(u-t)\,\mathrm{d}u \\ &= \lim_{0<\sigma\to 0} \int_{\mathbb{R}} s(u) \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi(u-t)\cdot\nu}\,\mathrm{e}^{-2(\pi\sigma\nu)^2}\,\mathrm{d}\nu\,\mathrm{d}u \\ &= \lim_{0<\sigma\to 0} \int_{\mathbb{R}} \mathrm{e}^{-2(\pi\sigma\nu)^2} \int_{\mathbb{R}} s(u)\,\mathrm{e}^{-\mathrm{i}2\pi(u-t)\cdot\nu}\,\mathrm{d}u\,\mathrm{d}\nu \\ &= \lim_{0<\sigma\to 0} \int_{\mathbb{R}} \mathrm{e}^{-2(\pi\sigma\nu)^2}\,\widehat{s}(\nu)\,\mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\,\mathrm{d}\nu \\ &= \int_{\mathbb{R}} \widehat{s}(\nu)\,\mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\,\mathrm{d}\nu. \end{aligned}$$

Inverse Fourier transform

We just found that the Fourier transform

$$\widehat{s}(\nu) = \int_{\mathbb{R}} s(t)\ \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ \mathrm{d}\nu. \tag{6}$$

has Fourier inverse transform

$$s(t) = \int_{\mathbb{R}} \widehat{s}(\nu)\ \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\ \mathrm{d}\nu. \tag{7}$$

By these formulas, we can present signals
as well in time t as in frequency ν,
whatever is most convenient!

Vector space of signals

Given signals $r, s : \mathbb{R} \to \mathbb{C}$ and scalar $\lambda \in \mathbb{C}$,
we obtain new signals $r + s, \lambda\, s : \mathbb{R} \to \mathbb{C}$ by

$$\begin{aligned} (r+s)(t) &= r(t) + s(t), \\ (\lambda\, s)(t) &= \lambda\, s(t). \end{aligned}$$

The space of "all signals" is a vector space,
with **inner product**

$$\langle r, s \rangle = \int_{\mathbb{R}} r(t)\, \overline{s(t)}\, \mathrm{d}t \ \in \ \mathbb{C},$$

and **norm**

$$\|s\| = \sqrt{\langle s, s \rangle} = \left(\int_{\mathbb{R}} |s(t)|^2\, \mathrm{d}t \right)^{1/2} \ \in \ \mathbb{R}^+.$$

Remember: energy is $\|s\|^2 = \langle s, s \rangle$.

Fourier transform preserves energy

Inner product between signals $r, s : \mathbb{R} \to \mathbb{C}$ is

$$\langle r, s \rangle := \int_{\mathbb{R}} r(t)\, \overline{s(t)}\, \mathrm{d}t \ \in \ \mathbb{C}.$$

Fourier transform preserves this inner product, because

$$\begin{aligned}
\langle \widehat{r}, \widehat{s} \rangle &= \int_{\mathbb{R}} \widehat{r}(\nu)\, \overline{\widehat{s}(\nu)}\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \widehat{r}(\nu) \overline{\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ s(t)\, \mathrm{d}t}\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\ \widehat{r}(\nu)\, \mathrm{d}\nu\ \overline{s(t)}\, \mathrm{d}t \\
&= \int_{\mathbb{R}} r(t)\, \overline{s(t)}\, \mathrm{d}t \quad = \quad \langle r, s \rangle.
\end{aligned}$$

Putting $r = s$, we see that Fourier transform preserves energy:

$$\|\widehat{s}\|^2 = \|s\|^2. \tag{8}$$

Symmetries of time and frequency

Time translation of signal $s:\mathbb{R}\to\mathbb{C}$ by time-lag $p\in\mathbb{R}$ is signal $T_p s:\mathbb{R}\to\mathbb{C}$, where

$$T_p s(t) := s(t-p). \tag{9}$$

Frequency modulation of $s:\mathbb{R}\to\mathbb{C}$ by frequency-lag $\alpha\in\mathbb{R}$ is signal $M_\alpha s:\mathbb{R}\to\mathbb{C}$, where

$$M_\alpha s(t) := \mathrm{e}^{+\mathrm{i}2\pi t\cdot\alpha} s(t). \tag{10}$$

After Fourier transforms: $\widehat{M_\alpha s} = T_\alpha \widehat{s}$ and $\widehat{T_p s} = M_{-p}\widehat{s}$, that is

$$\begin{aligned} \widehat{M_\alpha s}(\nu) &= T_\alpha \widehat{s}(\nu), \\ \widehat{T_p s}(\nu) &= M_{-p}\widehat{s}(\nu). \end{aligned}$$

Integral operators

We want to transform input signal $s : \mathbb{R} \to \mathbb{C}$
to output signal $Ls = L(s) : \mathbb{R} \to \mathbb{C}$.
Suppose L is *linear*, i.e.

$$\begin{aligned} L(r+s) &= L(r) + L(s) \quad \text{and} \\ L(\lambda s) &= \lambda\, L(s) \end{aligned}$$

for all signals $r, s : \mathbb{R} \to \mathbb{C}$ and constants $\lambda \in \mathbb{C}$.
Linear transform L presented as an **integral operator**:

$$Ls(t) = \int_{\mathbb{R}} K_L(t,u)\; s(u)\; \mathrm{d}u, \tag{11}$$

where K_L is the **kernel** of L.
Remark: integral operator L has "essentially unique" kernel K_L!

Time-invariant operators

Let operator L be **time-invariant**: $T_p L = L T_p$ for all $p \in \mathbb{R}$, i.e.

$$T_p L s(t) = L T_p s(t) \tag{12}$$

for all signals $s : \mathbb{R} \to \mathbb{C}$ and for all times $t, p \in \mathbb{R}$;
in other words, $L = T_{-p} L T_p$, which means

$$\begin{aligned}
\int_{\mathbb{R}} K_L(t,u)\, s(u)\, \mathrm{d}u &= Ls(t) = T_{-p} L T_p s(t) = L T_p s(t+p) \\
&= \int_{\mathbb{R}} K_L(t+p,u)\, T_p s(u)\, \mathrm{d}u \\
&= \int_{\mathbb{R}} K_L(t+p,u)\, s(u-p)\, \mathrm{d}u \\
&= \int_{\mathbb{R}} K_L(t+p,u+p)\, s(u)\, \mathrm{d}u.
\end{aligned}$$

Thus $K_L(t,u) = K_L(t+p,u+p)$ for all $p,t,u \in \mathbb{R}$, especially $K_L(t,u) = K_L(t-u,0) = r(t-u)$ for some signal $r : \mathbb{R} \to \mathbb{C}$...

Convolution

Convolution of signals $r, s : \mathbb{R} \to \mathbb{C}$ is signal $r * s : \mathbb{R} \to \mathbb{C}$ given by

$$r * s(t) := \int_{\mathbb{R}} r(t-u)\, s(u)\, \mathrm{d}u. \tag{13}$$

Remark: time-invariant operator L is of form $Ls(t) = r * s(t)$, where its kernel satisfies $K_L(t,u) = r(t-u)$.
Exercise: show that

$$\widehat{r * s} = \widehat{r}\, \widehat{s}, \tag{14}$$

that is

$$\widehat{r * s}(\nu) = \widehat{r}(\nu)\, \widehat{s}(\nu).$$

Thus "convolution in time" is "multiplication in frequency".
This is one of the most important properties in Fourier analysis!

Convolution smoothing

$r * s$ is smooth if r is smooth:

$$(r * s)' = r' * s, \tag{15}$$

because

$$(r*s)'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} r(t-u)\, s(u)\, \mathrm{d}u = \int_{\mathbb{R}} r'(t-u)\, s(u)\, \mathrm{d}u = r'*s(t).$$

Example:

$$\varphi_\sigma(t) := \frac{1}{\sqrt{2\pi}\,\sigma}\, \mathrm{e}^{-(t/\sigma)^2/2} \quad \Longrightarrow \quad \widehat{\varphi_\sigma}(\nu) = \mathrm{e}^{-2(\pi\sigma\nu)^2}$$

$$\begin{aligned} \Longrightarrow \quad \widehat{\varphi_\sigma * s}(t) \quad &= \quad \widehat{\varphi_\sigma}(\nu)\, \widehat{s}(\nu) \\ &= \quad \mathrm{e}^{-2(\pi\sigma\nu)^2}\, \widehat{s}(\nu) \\ &\xrightarrow{0<\sigma\to 0} \quad \widehat{s}(\nu). \end{aligned}$$

Dirac delta signal δ_p at time $p \in \mathbb{R}$ satisfies

$$\int_{\mathbb{R}} s(t)\ \delta_p(t)\ \mathrm{d}t = s(p) \tag{16}$$

for all continuous signals $s : \mathbb{R} \to \mathbb{C}$.
Interpretation: Dirac delta δ_p is the sudden unit impulse at time p,

$$\delta_p = \lim_{0<\sigma\to 0} \varphi_{p,\sigma}.$$

Fourier transform of Dirac delta:

$$\widehat{\delta_p}(\nu) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ \delta_p(t)\ \mathrm{d}t = \mathrm{e}^{-\mathrm{i}2\pi p\cdot\nu}.$$

Remark: Energy

$$\|\delta_p\|^2 = \|\widehat{\delta_p}\|^2 = \int_{\mathbb{R}} |\mathrm{e}^{-\mathrm{i}2\pi p\cdot\nu}|^2\ \mathrm{d}\nu = \int_{\mathbb{R}} 1\ \mathrm{d}\nu = \infty.$$

δ_p cannot be a function in any ordinary sense! We may think that $\delta_p(t) = 0$ if $t \neq p$, but writing $\delta_p(p) = \infty$ would be dubious! Such a weird entity δ_p is called a **(tempered) distribution**.

Fourier meets Dirac:

$$\begin{aligned}
s(t) &= \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\, \widehat{s}(\nu)\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\, s(u)\, \mathrm{d}u\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\, \mathrm{d}\nu \right] s(u)\, \mathrm{d}u \\
&= \int_{\mathbb{R}} \delta_0(t-u)\, s(u)\, \mathrm{d}u \\
&= \delta_0 * s(t),
\end{aligned}$$

as it should be. More generally, $\delta_p * s = T_p s$:

$$\begin{aligned}
\delta_p * s(t) &= \int_{\mathbb{R}} \delta_p(t-u)\, s(u)\, \mathrm{d}u \\
&= s(t-p) \;=\; T_p s(t).
\end{aligned}$$

Fourier integral in dimension $d \in \mathbb{Z}^+ = \{1, 2, 3, 4, 5, \cdots\}$

Fourier transform $\widehat{s} : \mathbb{R}^d \to \mathbb{C}$ for signal $s : \mathbb{R}^d \to \mathbb{C}$ is given by

$$\widehat{s}(\nu) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ s(t)\ \mathrm{d}t, \qquad (17)$$

where $t = (t_1, \cdots, t_d) \in \mathbb{R}^d$, $\nu = (\nu_1, \cdots, \nu_d) \in \mathbb{R}^d$,
$t \cdot \nu = \sum_{k=1}^{d} t_k \cdot \nu_k = t_1\nu_1 + \cdots + t_d\nu_d \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \cdots\ \mathrm{d}t = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \cdots\ \mathrm{d}t_1\ \cdots\ \mathrm{d}t_d.$$

Energy $\|s\|^2 := \int_{\mathbb{R}^d} |s(t)|^2\, \mathrm{d}t$, and for example

$$\begin{aligned} s(t) &= \int_{\mathbb{R}^d} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\ \widehat{s}(\nu)\ \mathrm{d}\nu, \\ \|s\|^2 &= \|\widehat{s}\|^2, \\ r * s(t) &:= \int_{\mathbb{R}^d} r(t-u)\ s(u)\ \mathrm{d}u, \\ \widehat{r * s}(\nu) &= \widehat{r}(\nu)\ \widehat{s}(\nu). \end{aligned}$$

Analog periodic world (A1)

Signal $s : \mathbb{R} \to \mathbb{C}$ is p-**periodic** if $T_p s = s$, meaning $s(t-p) = s(t)$ for all $t \in \mathbb{R}$: in this case, we denote $s : \mathbb{R}/p\mathbb{Z} \to \mathbb{C}$. Without losing generality, we deal with 1-periodic signals

$$s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$$

for which $s(t-1) = s(t)$ for all $t \in \mathbb{R}$; then the **Fourier coefficient transform** $\mathcal{F}_{\mathbb{R}/\mathbb{Z}} s = \widehat{s} : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\widehat{s}(\nu) := \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\, s(t)\, \mathrm{d}t = \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\, s(t)\, \mathrm{d}t. \tag{18}$$

Exercise: show that $\widehat{s}(\nu) = c_\nu \in \mathbb{C}$, when $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is given by

$$s(t) := \sum_{k\in\mathbb{Z}} c_k\, \mathrm{e}^{\mathrm{i}2\pi t\cdot k} = \sum_{k=-\infty}^{\infty} c_k\, \mathrm{e}^{\mathrm{i}2\pi t\cdot k}$$

(naturally, provided that signal s is "nice enough").

For periodic signal $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, Fourier coefficients

$$\widehat{s}(\nu) = \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu} s(t)\,\mathrm{d}t \quad \in \quad \mathbb{C}.$$

It turns out that "nice enough" $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ can be recovered from its Fourier coefficients by **Fourier series**

$$s(t) = \sum_{\nu\in\mathbb{Z}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\,\widehat{s}(\nu) = \sum_{\nu=-\infty}^{+\infty} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\,\widehat{s}(\nu). \tag{19}$$

Thus **periodic analog signal** $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ has the same information content as **non-periodic digital signal** $\widehat{s} : \mathbb{Z} \to \mathbb{C}$; using the signal classification presented in the beginning of the course, this means that classes (A1) and (D0) are dual to each other by Fourier transform, so that properties in (A1) have corresponding "mirrored" properties in (D0), and vice versa.

(A1) Where do Fourier series come from?

Poisson kernel φ_r, for which $0 < \varphi_r(t) < \infty$ and $\int_0^1 \varphi_r(t)\ \mathrm{d}t = 1$:

$$\varphi_r(t) := \sum_{\nu \in \mathbb{Z}} r^{|\nu|}\ \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu} = \frac{1-r^2}{1+r^2-2r\cos(2\pi t)}, \qquad (20)$$

where $0 < r < 1$. Then for smooth $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ we have

$$\begin{aligned} s(t) &= \lim_{r\to 1^-} \int_0^1 s(u)\ \varphi_r(t-u)\ \mathrm{d}u \\ &= \lim_{r\to 1^-} \int_0^1 s(u) \sum_{\nu\in\mathbb{Z}} r^{|\nu|}\ \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ \mathrm{d}u \\ &= \lim_{r\to 1^-} \sum_{\nu\in\mathbb{Z}} \widehat{s}(\nu)\ r^{|\nu|}\ \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu} \\ &= \sum_{\nu\in\mathbb{Z}} \widehat{s}(\nu)\ \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu}. \end{aligned}$$

Energy conservation in Fourier coefficients and series

Let $r, s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, so that $\widehat{r}, \widehat{s} : \mathbb{Z} \to \mathbb{C}$. Then

$$\begin{aligned}
\langle \widehat{r}, \widehat{s} \rangle &:= \sum_{\nu \in \mathbb{Z}} \widehat{r}(\nu)\, \overline{\widehat{s}(\nu)} \\
&= \sum_{\nu \in \mathbb{Z}} \widehat{r}(\nu)\, \overline{\int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu}\ s(t)\, \mathrm{d}t\, \mathrm{d}\nu} \\
&= \int_{\mathbb{R}/\mathbb{Z}} \sum_{\nu \in \mathbb{Z}} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \nu}\ \widehat{r}(\nu)\ \overline{s(t)}\, \mathrm{d}t \\
&= \int_{\mathbb{R}/\mathbb{Z}} r(t)\ \overline{s(t)}\, \mathrm{d}t \quad =: \quad \langle r, s \rangle.
\end{aligned}$$

We see that Fourier coefficient/series transform preserves energy

$$\|s\|^2 := \langle s, s \rangle = \langle \widehat{s}, \widehat{s} \rangle =: \|\widehat{s}\|^2. \tag{21}$$

(A1) Convolution of periodic signals

Convolution $r * s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ of periodic signals $r, s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is defined by

$$r * s(t) := \int_{\mathbb{R}/\mathbb{Z}} r(t-u)\, s(u)\, \mathrm{d}u. \tag{22}$$

By easy computation, we see that $\widehat{r * s} = \widehat{r}\,\widehat{s}$:

$$\widehat{r * s}(\nu) = \widehat{r}(\nu)\, \widehat{s}(\nu).$$

Naturally, periodic convolution has smoothing properties:

$$(r * s)'(t) = r' * s(t).$$

Thus, convolution works in similar manner for both periodic and non-periodic signals!

Periodization and Poisson summation formula

Periodization of signal $s : \mathbb{R} \to \mathbb{C}$ is $\mathcal{P}s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, where

$$\mathcal{P}s(t) := \sum_{k \in \mathbb{Z}} s(t-k).$$

Their Fourier transforms $\widehat{s} : \mathbb{R} \to \mathbb{C}$ and $\widehat{\mathcal{P}s} : \mathbb{Z} \to \mathbb{C}$ satisfy

$$\begin{aligned} \widehat{\mathcal{P}s}(\nu) &= \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu} \sum_{k \in \mathbb{Z}} s(t-k)\,\mathrm{d}t \\ &= \sum_{k \in \mathbb{Z}} \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi (t-k) \cdot \nu}\, s(t-k)\,\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu}\, s(t)\,\mathrm{d}t \;=\; \widehat{s}(\nu). \end{aligned}$$

Result $\widehat{\mathcal{P}s}(\nu) = \widehat{s}(\nu)$ yields **Poisson summation formula**

$$\sum_{\nu \in \mathbb{Z}} \widehat{s}(\nu) \;=\; \overset{\textit{Exercise}}{\cdots} \;=\; \sum_{k \in \mathbb{Z}} s(k).$$

Digital non-periodic world (D0), or DTFT

Fourier transform of digital signal $s : \mathbb{Z} \to \mathbb{C}$ is periodic signal $\mathcal{F}_{\mathbb{Z}}(s) = \widehat{s} : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ defined by

$$\widehat{s}(\nu) := \sum_{t \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu} \; s(t). \tag{23}$$

This is called **Discrete Time Fourier Transform** (DTFT).
Remark: this is essentially similar to the previous Fourier series case (apart from the sign of the imaginary unit i).
For digital signals $r, s : \mathbb{Z} \to \mathbb{C}$, we define the convolution $r * s : \mathbb{Z} \to \mathbb{C}$ by

$$r * s(t) := \sum_{u \in \mathbb{Z}} r(t-u) \; s(u). \tag{24}$$

The reader may check that $\widehat{r * s} = \widehat{r}\,\widehat{s}$, that is

$$\widehat{r * s}(\nu) = \widehat{r}(\nu) \; \widehat{s}(\nu).$$

Inverse transform to DTFT

For $s : \mathbb{Z} \to \mathbb{C}$ we have DTFT $\widehat{s} : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, where

$$\widehat{s}(\nu) := \sum_{t \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu}\ s(t).$$

The inverse transform is verified by a direct calculation:

$$\begin{aligned}
\int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \nu}\ \widehat{s}(\nu)\ \mathrm{d}\nu &= \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \nu} \sum_{u \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi u \cdot \nu} s(u)\ \mathrm{d}\nu \\
&= \sum_{u \in \mathbb{Z}} s(u) \int_0^1 \mathrm{e}^{\mathrm{i}2\pi (t-u) \cdot \nu}\ \mathrm{d}\nu \\
&= s(t).
\end{aligned}$$

Well, no wonder: this is just because signal classes (A0) and (D1) are dual to each other by Fourier transform! Thus, no need to check conservation of energy again.

From Poisson summation to sampling

Poisson summation formula $\sum_{\nu\in\mathbb{Z}} \widehat{s}(\nu) = \sum_{k\in\mathbb{Z}} s(k)$ is equivalent to

$$\sum_{\alpha\in\mathbb{Z}} \widehat{s}(\nu-\alpha) = \sum_{k\in\mathbb{Z}} s(k)\ \mathrm{e}^{-\mathrm{i}2\pi k\cdot\nu}. \tag{25}$$

Now suppose that $\widehat{s_1}(\nu) = 0$ whenever $|\nu| \geq 1/2$: then

$$\begin{aligned}
\widehat{s_1}(\nu) &= \mathbf{1}_{]-1/2,+1/2[}(\nu) \sum_{\alpha\in\mathbb{Z}} \widehat{s_1}(\nu-\alpha) \\
&\overset{(25)}{=} \mathbf{1}_{]-1/2,+1/2[}(\nu) \sum_{k\in\mathbb{Z}} s_1(k)\ \mathrm{e}^{-\mathrm{i}2\pi k\cdot\nu} \\
&= \sum_{k\in\mathbb{Z}} s_1(k)\ \mathrm{e}^{-\mathrm{i}2\pi k\cdot\nu}\ \mathbf{1}_{]-1/2,+1/2[}(\nu),
\end{aligned}$$

leading to **normalized Whittaker–Shannon sampling formula**

$$s_1(t) = \sum_{k\in\mathbb{Z}} s_1(k)\ \mathrm{sinc}(t-k). \tag{26}$$

Nyquist–Shannon sampling theorem

... From this, we get **Whittaker–Shannon sampling formula**

$$s(t) = \sum_{k \in \mathbb{Z}} s(\frac{k}{2B}) \operatorname{sinc}(2Bt - k), \tag{27}$$

which is valid if $\widehat{s}(\nu) = 0$ whenever $|\nu| \geq B$.
Related to this formula, **Nyquist–Shannon sampling theorem** says: If analog signal $s : \mathbb{R} \to \mathbb{C}$ is **band-limited** (meaning $\widehat{s}(\nu) = 0$ whenever $|\nu| \geq B$), then we are able to reconstruct it from its equispaced sampled values, i.e. from the corresponding digital signal $r : \mathbb{Z} \to \mathbb{C}$, where

$$r(k) := s(k/(2B)).$$

In other words, Whittaker–Shannon formula builds a bridge between non-periodic analog signals and non-periodic digital signals!

Digital periodic world (D1), or DFT

N-periodic digital signal $s : \mathbb{Z} \to \mathbb{C}$ satisfies $s(t - N) = s(t)$ for all $t \in \mathbb{Z}$: then we denote

$$s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}. \tag{28}$$

Its **discrete Fourier transform** (DFT) $\widehat{s} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is defined by

$$\widehat{s}(\nu) := \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu / N} \, s(t). \tag{29}$$

Notice that in the exponential we have $t \cdot \nu / N$ instead of $t \cdot \nu$. Exercise: show that the inverse $\widehat{s} \mapsto s$ of DFT is given by

$$s(t) = \frac{1}{N} \sum_{\nu=1}^{N} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \nu / N} \, \widehat{s}(\nu). \tag{30}$$

Notice the factor $\frac{1}{N}$ in this formula!

Energy and convolution

Exercise: defining here **energy** $\|s\|^2 := \sum_{t=1}^{n} |s(t)|^2$, find constant c_N such that for all signals $s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$

$$\|s\|^2 = c_N \|\widehat{s}\|^2. \tag{31}$$

Hence "energy is conserved up to a constant".
For digital signals $r, s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, we define the discrete **convolution** $r * s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by

$$r * s(t) := \sum_{u=1}^{N} r(t-u)\, s(u). \tag{32}$$

The reader may check that $\widehat{r * s} = \widehat{r}\,\widehat{s}$, that is

$$\widehat{r * s}(\nu) = \widehat{r}(\nu)\, \widehat{s}(\nu).$$

DFT related to DTFT (D1 vs. D0)

For "nice" non-periodic $s : \mathbb{Z} \to \mathbb{C}$, define $s_N : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by

$$s_N(t) := \sum_{k \in \mathbb{Z}} s(t - kN).$$

Then $\widehat{s_N} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is naturally related to $\widehat{s} : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$:

$$\begin{aligned}
\widehat{s_N}(\nu) &= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu / N} \, s_N(t) \\
&= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu / N} \sum_{k \in \mathbb{Z}} s(t - kN) \\
&= \sum_{u \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi u \cdot \nu / N} \, s(u) \;=\; \widehat{s}(\nu / N).
\end{aligned}$$

Hence $\widehat{s_N}(\nu) = \widehat{s}(\nu/N)$ for all ν.

DFT related to Fourier series/coefficients (D1 vs. A1)

For "nice" periodic $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, define $s_N : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by

$$s_N(t) := s(t/N).$$

Then $\widehat{s_N} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is naturally related to $\widehat{s} : \mathbb{Z} \to \mathbb{C}$:

$$\begin{aligned}
\widehat{s_N}(\nu) &= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\ s(t/N) \\
&= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N} \sum_{\alpha\in\mathbb{Z}} \widehat{s}(\alpha)\ \mathrm{e}^{+\mathrm{i}2\pi(t/N)\cdot\alpha} \\
&= \sum_{\alpha\in\mathbb{Z}} \widehat{s}(\alpha) \sum_{t=1}^{N} \mathrm{e}^{\mathrm{i}2\pi t\cdot(\alpha-\nu)/N} \ = \ N \sum_{k\in\mathbb{Z}} \widehat{s}(\nu - kN).
\end{aligned}$$

Hence $\widehat{s_N}(\nu) = N \sum_{k\in\mathbb{Z}} \widehat{s}(\nu - kN)$ for all $\nu \in \mathbb{Z}$.

FFT (Fast Fourier Transform)...

FFT (Fast Fourier Transform) is a fast method for computing DFT. It is a divide-and-conquer algorithm, one of the most important tools in engineering and applied mathematics. Idea: Given signal $s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, we want to find $F_N s = \widehat{s} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, where $N = 2^k$. Split computation into two smaller size DFTs:

$$\begin{aligned}
F_N s(\nu) &= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\, s(t) \\
&= \sum_{t\in\{1,3,5,\cdots,N-1\}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\, s(t) + \sum_{t\in\{2,4,6,\cdots,N\}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\, s(t) \\
&= \sum_{t=1}^{N/2} \mathrm{e}^{-\mathrm{i}2\pi(2t-1)\cdot\nu/N} s(2t-1) + \sum_{t=1}^{N/2} \mathrm{e}^{-\mathrm{i}2\pi(2t)\cdot\nu/N} s(2t) \\
&= \mathrm{e}^{+\mathrm{i}2\pi\nu/N}\, F_{N/2} s_{\mathrm{Odd}}(\nu) + F_{N/2} s_{\mathrm{Even}}(\nu).
\end{aligned}$$

Hence we just need to calculate $F_{N/2} s_{\mathrm{Odd}}$ and $F_{N/2} s_{\mathrm{Even}}$...

Why FFT requires only about $N\log(N)$ time units?

We say that the **complexity** of algorithm F_N is the "essential number" M_N of multiplications needed in computation. Obviously

$$F_N s(\nu) = \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\, s(t)$$

yields $M_1 = 1$ and $M_N \le N^2$. However,

$$F_N s(\nu) = \mathrm{e}^{+\mathrm{i}2\pi\nu/N}\, F_{N/2} s_{\mathrm{Odd}}(\nu) + F_{N/2} s_{\mathrm{Even}}(\nu) \tag{33}$$

implies recursively

$$\begin{aligned}
M_N &\overset{(33)}{\le} N + 2\,M_{N/2} \\
&\overset{(33)}{\le} N + 2(N/2 + 2\,M_{N/4}) \;=\; 2\,N + 4\,M_{N/4} \\
&\overset{(33)}{\le} 2\,N + 4(N/4 + 2\,M_{N/8}) \;=\; 3\,N + 8\,M_{N/8} \\
\cdots &\overset{(33)}{\le} \log_2(N)\,N + N\,M_{N/N} \;=\; N\log(N) + N \;\approx\; N\log(N).
\end{aligned}$$

Fast convolution via FFT

Direct calculation of discrete convolution $r * s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ of signals $r, s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ would require N^2 multiplications, as

$$r * s(t) = \sum_{u=1}^{N} r(t-u)\, s(u).$$

However,

$$\widehat{r * s}(\nu) = \widehat{r}(\nu)\, \widehat{s}(\nu),$$

where finding $\widehat{r}\,\widehat{s}$ takes only N multiplications. Computing each of

$$r \mapsto \widehat{r}, \quad s \mapsto \widehat{s}, \quad \widehat{r}\widehat{s} \mapsto r * s$$

takes only about $N \log(N)$ multiplications by FFT. Thus, computation $(r, s) \mapsto r * s$ has essential complexity $N \log(N)$, too!

Matlab computing FFT

`Matlab` command `fft` (Fast Fourier Transform) works as follows: vector $\mathtt{X} = \mathtt{fft}(\mathtt{x})$ for vector $\mathtt{x} = [\mathtt{x}(1)\ \mathtt{x}(2)\ \ldots\ \mathtt{x}(\mathtt{N})]$ is given by

$$\mathtt{X}(\mathtt{m}) = \sum_{\mathtt{k}=1}^{\mathtt{N}} \mathrm{e}^{-\mathrm{i}2\pi(\mathtt{k}-1)(\mathtt{m}-1)/\mathtt{N}}\ \mathtt{x}(\mathtt{k}), \tag{34}$$

instead of our more natural definition

$$\widehat{\mathtt{x}}(m) := \sum_{\mathtt{k}=1}^{\mathtt{N}} \mathrm{e}^{-\mathrm{i}2\pi\mathtt{k}\cdot\mathtt{m}/N}\ \mathtt{x}(\mathtt{k}). \tag{35}$$

That is, `Matlab` shifts both time and frequency by 1 always, and such a weird definition does not match well e.g. with convolution!
So, you have been warned!!!
Otherwise, `Matlab` is fine for computational Fourier analysis.

Time-frequency analysis

Next we try to understand behavior of signals simultaneously in both time and frequency. Applications of such time-frequency analysis include audio signal processing (phonetics, treating speech defects, speech synthesis, analyzing animal sounds, music), medical visualizations of EEG and ECG (ElectroEncephaloGraphy and ElectroCardioGraphy), sonar and radar imaging, seismology, quantum physics etc.
A time-frequency distribution for signal $s : \mathbb{R} \to \mathbb{C}$ is typically

$$As : \mathbb{R} \times \mathbb{R} \to \mathbb{C},$$

where $As(t, \nu)$ is "intensity of s at time-frequency (t, ν)".
There are many different time-frequency distributions to choose from, notably members of Leon Cohen's class, which includes e.g. all spectrograms and so-called Born–Jordan distribution.

Windowed Fourier Transform (STFT, Short-Time Fourier Transform)

For signals $s, w : \mathbb{R} \to \mathbb{C}$, w-**windowed Fourier transform** (**STFT**, Short-Time Fourier Transform) $F(s,w) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is

$$F(s,w)(t,\nu) := \widehat{s\,\overline{w_t}}(\nu), \tag{36}$$

where $w_t(u) = w(u-t)$. That is,

$$F(s,w)(t,\nu) = \int_{\mathbb{R}} s(u)\,\overline{w(u-t)}\,\mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu}\,\mathrm{d}u.$$

Idea: Fourier transform $\widehat{s}(\nu)$ measures "content" of s at frequency $\nu \in \mathbb{R}$ over all times. $F(s,w)(t,\nu)$ measures "content" of s at time-frequency $(t,\nu) \in \mathbb{R} \times \mathbb{R}$ (when viewing s through window w).

Spectrogram (Sonogram)

Spectrogram related to the w-windowed Fourier transform is

$$|F(s,w)|^2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+. \tag{37}$$

Idea: $|F(s,w)(t,\nu)|^2 \geq 0$ is the "energy intensity" of signal $s : \mathbb{R} \to \mathbb{C}$ at time-frequency $(t,\nu) \in \mathbb{R} \times \mathbb{R}$ (when viewing s through window w).
For signal $s : \mathbb{R} \to \mathbb{C}$, choosing window w influences heavily the corresponding w-STFT and w-spectrogram!
In `Matlab`, try experimenting:

```
help spectrogram
```

Or, program your own spectrogram as in Exercises, implementing

$$|F(s,w)(t,\nu)|^2 = \left| \int_{\mathbb{R}} s(u)\, \overline{w(u-t)}\, \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu}\, \mathrm{d}u \right|^2.$$

Born–Jordan time-frequency distribution

For signals $r, s : \mathbb{R} \to \mathbb{C}$, the **Born–Jordan transform** $Q(r,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is defined by

$$\begin{aligned} Q(r,s)(t,\nu) &:= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu} \, \frac{1}{u} \int_{t-u/2}^{t+u/2} r(z+u/2)\,\overline{s(z-u/2)}\,\mathrm{d}z\,\mathrm{d}u \\ &= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu} \, \frac{1}{u} \int_{t}^{t+u} r(z)\,\overline{s(z-u)}\,\mathrm{d}z\,\mathrm{d}u. \end{aligned}$$

The **Born–Jordan distribution** of $s : \mathbb{R} \to \mathbb{C}$ is

$$Qs = Q(s) := Q(s,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}. \tag{38}$$

Interpretation: $Qs(t,\nu) \in \mathbb{R}$ is the "energy intensity" of $s : \mathbb{R} \to \mathbb{C}$ at time-frequency $(t,\nu) \in \mathbb{R} \times \mathbb{R}$.

Properties of Born–Jordan distribution

Marginals: $\int_{\mathbb{R}} Qs(t,\nu)\,\mathrm{d}t = |\widehat{s}(\nu)|^2$, $\int_{\mathbb{R}} Qs(t,\nu)\,\mathrm{d}\nu = |s(t)|^2$.

Thus energy $\int_{\mathbb{R}}\int_{\mathbb{R}} Qs(t,\nu)\,\mathrm{d}t\,\mathrm{d}\nu = \|s\|^2$.

Natural Fourier symmetries: $Q\widehat{s}(\nu,t) = Qs(-t,\nu)$.

If $r(t) := s(t-t_0)$ and $q(t) := \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu_0}\,s(t)$ then

$$\begin{aligned} Qr(t,\nu) &= Qs(t-t_0,\nu), \\ Qq(t,\nu) &= Qs(t,\nu-\nu_0). \end{aligned}$$

$Q\delta_{t_0}(t,\nu) = \delta_{t_0}(t)$.

$Qe_{\nu_0}(t,\nu) = \delta_{\nu_0}(\nu)$, where $e_{\nu_0}(t) := \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu_0}$.

For $\alpha < \beta$: $Q(\lambda e_\alpha + \mu e_\beta)(t,\nu) =$

$$|\lambda|^2\delta_\alpha(\nu) + |\mu|^2\delta_\beta(\nu) + 2\,\mathrm{Re}\left(\lambda\,\overline{\mu}\,e_{\alpha-\beta}(t)\right)\frac{\mathbf{1}_{[\alpha,\beta]}(\nu)}{\beta-\alpha}.$$

A **time-frequency symbol** is function $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. Now we design an integral operator L_σ such that we get a "best possible Born–Jordan approximation"

$$Q(L_\sigma s)(t,\nu) \approx \sigma(t,\nu)\, Qs(t,\nu)$$

for all signals $s : \mathbb{R} \to \mathbb{C}$ and for all $(t,\nu) \in \mathbb{R} \times \mathbb{R}$. Namely,

$$\langle r, L_\sigma s \rangle = \langle Q(r,s), \sigma \rangle \tag{39}$$

for all signals $r, s : \mathbb{R} \to \mathbb{C}$: here $\langle r, L_\sigma s \rangle = \langle Q(r,s), \sigma \rangle =$

$$\begin{aligned}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} Q(r,s)(z,\nu)\, \overline{\sigma(z,\nu)}\, \mathrm{d}z\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi w \cdot \nu} \frac{1}{w} \int_{z-\frac{w}{2}}^{z+\frac{w}{2}} r(\tilde{t} + \frac{w}{2}) \overline{s(\tilde{t} - \frac{w}{2})}\, \mathrm{d}\tilde{t}\, \mathrm{d}w\, \overline{\sigma(z,\nu)}\, \mathrm{d}z\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} r(t) \left[\int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu} s(u) \frac{1}{u-t} \int_t^u \sigma(z,\nu)\, \mathrm{d}z\, \mathrm{d}u\, \mathrm{d}\nu \right]^* \mathrm{d}t.
\end{aligned}$$

... Born–Jordan filter design

... Hence

$$L_\sigma s(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu} s(u)\ a(t,u,\nu)\ \mathrm{d}u\ \mathrm{d}\nu, \qquad (40)$$

where $a(t,t,\nu) = \sigma(t,\nu)$, and for $t \neq u$ we have amplitude

$$a(t,u,\nu) = \frac{1}{u-t} \int_t^u \sigma(z,\nu)\ \mathrm{d}z. \qquad (41)$$

We obtained

$$L_\sigma s(t) = \int_{\mathbb{R}} K_{L_\sigma}(t,u)\ s(u)\ \mathrm{d}u,$$

where kernel $K_{L_\sigma} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ of integral operator L_σ is given by $K_{L_\sigma}(t,t) = \int_{\mathbb{R}} \sigma(t,\nu)\ \mathrm{d}\nu$, and for $t \neq u$ by

$$K_{L_\sigma}(t,u) = \frac{1}{u-t} \int_t^u \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ \sigma(z,\nu)\ \mathrm{d}\nu\ \mathrm{d}z. \qquad (42)$$

Filtering examples

On previous page, suppose time-invariance $\sigma(t,\nu) = \widehat{\psi}(\nu)$ for all $(t,\nu) \in \mathbb{R} \times \mathbb{R}$. Naturally, then $L_\sigma s = \psi * s$, because

$$\begin{aligned} L_\sigma s(t) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ s(u)\ \frac{1}{u-t} \int_t^u \widehat{\psi}(\nu)\ \mathrm{d}z\ \mathrm{d}u\ \mathrm{d}\nu \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ s(u)\ \widehat{\psi}(\nu)\ \mathrm{d}u\ \mathrm{d}\nu \\ &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu}\ \widehat{s}(\nu)\ \widehat{\psi}(\nu)\ \mathrm{d}\nu \ = \ \psi * s(t). \end{aligned}$$

On previous page, suppose frequency-invariance $\sigma(t,\nu) = \varphi(t)$ for all $(t,\nu) \in \mathbb{R} \times \mathbb{R}$. Then $a(t,u,\nu) = b(t,u)$ so that $L_\sigma s = \varphi\, s$:

$$\begin{aligned} L_\sigma s(t) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ s(u)\ b(t,u)\ \mathrm{d}u\ \mathrm{d}\nu \\ &= s(t)\ b(t,t) \ = \ \varphi(t)\, s(t). \end{aligned}$$

Time-limited signal which is band-limited, too?

Let $\|s\|^2 < \infty$, where $s : \mathbb{R} \to \mathbb{C}$ is limited in time-frequency:

$$s(t) = 0 = \widehat{s}(\nu)$$

whenever $|t| > M$ and $|\nu| > M$ for some constant $M < \infty$.
Then define analytic function $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) := \int_{-M}^{M} \mathrm{e}^{-\mathrm{i}2\pi t \cdot z} \; s(t) \; \mathrm{d}t.$$

Due to analyticity, for any $a \in \mathbb{C}$ we have power series

$$h(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \; h^{(k)}(a) \; (z-a)^k.$$

If $M < a \in \mathbb{R}$ then $h(a) = \widehat{s}(a) = 0$, yielding $h(z) \equiv 0$ for all $z \in \mathbb{C}$.
But here $\widehat{s}(\nu) = h(\nu) \equiv 0$ for all $\nu \in \mathbb{R}$, so $s(t) \equiv 0$ for all $t \in \mathbb{R}$.
[Remark: Schwartz test functions $s \in \mathcal{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$
(e.g. Gaussian signals) are "almost time- and frequency-limited",
because $s(t), \widehat{s}(t) \to 0$ rapidly as $|t| \to \infty$.]

Heat flow: historical origin of Fourier analysis

Let $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfy so-called **heat equation**

$$\frac{\partial}{\partial t} u(x,t) = \alpha \left(\frac{\partial}{\partial x} \right)^2 u(x,t), \qquad (43)$$

with initial condition $u(x,0) = f(x)$, where $\alpha > 0$ is the thermal diffusivity constant. Here $u_t(x) = u(x,t)$ is "temperature at point x at time t". Taking Fourier transform in the x-variable, we get

$$\frac{\partial}{\partial t} \widehat{u_t}(\xi) = -(2\pi\xi)^2 \alpha \; \widehat{u_t}(\xi) \quad \text{and} \quad \widehat{u_0}(\xi) = \widehat{f}(\xi),$$

so that

$$\begin{aligned} \widehat{u_t}(\xi) &= \mathrm{e}^{-(2\pi\xi)^2 \alpha t} \; \widehat{f}(\xi), \\ u(x,t) &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi x \cdot \xi} \; \mathrm{e}^{-(2\pi\xi)^2 \alpha t} \; \widehat{f}(\xi) \, \mathrm{d}\xi. \end{aligned}$$

Fourier found this reasoning for periodic x case in 1807, but already Daniel Bernoulli and Leonhard Euler considered vibrating strings as trigonometric series in 1753; and Gauss invented FFT in 1805.

Review: how are different Fourier transforms related?

Time space G (continuous $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$; discrete $\mathbb{Z}$ and $\mathbb{Z}/N\mathbb{Z}$).
Frequency space $\widehat{G}$ is dual to the time space G.
Signal $s : G \to \mathbb{C}$ has Fourier transform $\widehat{s} : \widehat{G} \to \mathbb{C}$,

$$\begin{aligned} \widehat{s}(\nu) &= \int_G \mathrm{e}^{-\mathrm{i}\langle t,\nu\rangle}\, s(t)\, \mathrm{d}t, \\ s(t) &= \int_{\widehat{G}} \mathrm{e}^{+\mathrm{i}\langle t,\nu\rangle}\, \widehat{s}(\nu)\, \mathrm{d}\nu, \end{aligned}$$

"energy conservation" $E(\widehat{s}) = E(s)$ (except for DFT), where

$$E(s) = \|s\|^2 = \int_G |s(t)|^2\, \mathrm{d}t$$

(for DFT, energy conservation needed a constant...).
Convolution $r * s : G \to \mathbb{C}$ of signals $r, s : G \to \mathbb{C}$,

$$r * s(t) = \int_G r(t-u)\, s(u)\, \mathrm{d}u,$$

which can be in finite case computed efficiently by FFT.

Review problems and questions

In your field of science/engineering, find examples of signals

$$s : \mathbb{R} \to \mathbb{C}, \qquad s : \mathbb{R}/\mathbb{Z} \to \mathbb{C},$$
$$s : \mathbb{Z} \to \mathbb{C}, \qquad s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}.$$

In each of these cases:

- How is Fourier transform defined? Which kind of signal is it?
- How is energy defined? Interpretation of energy?
- How does the inverse Fourier transform look like?
- How is convolution defined? Applications to signal processing?

How are these different Fourier transforms related to each other?
Why is FFT fast?
What do time-frequency distributions tell us?