1.3 Integration

In this section, let $(X, \mathcal{M}, \mu)$ be a complete measure space. The *μ-integral*

$$\int f \, \mathrm{d}\mu$$

of an $\mathcal{M}$-measurable function $f : X \to [-\infty, +\infty]$ is defined step-by-step:

1. First for a simple function.
2. Then for a non-negative function.
3. Finally, the general definition.

Definition 1.3.1. Let $s : X \to [0, \infty[$ be an $\mathcal{M}$-measurable simple function. Its *integral* $\int s \mathrm{d}\mu \in [0, \infty]$ is

$$\int s \, \mathrm{d}\mu = \int \sum_{a \in s(X)} a \, \chi_{\{s=a\}} \, \mathrm{d}\mu := \sum_{a \in s(X)} a \cdot \mu(\{s = a\}), \tag{1.28}$$

with the convention $0 \cdot \infty := 0$. Especially, $\int \chi_E \mathrm{d}\mu = \mu(E)$ for $E \in \mathcal{M}$.

Definition 1.3.2. Let $f^+ : X \to [0, \infty]$ be an $\mathcal{M}$-measurable non-negative function. Its *integral* $\int f^+ \mathrm{d}\mu \in [0, \infty]$ is

$$\int f^+ \mathrm{d}\mu := \sup \left\{ \int s \, \mathrm{d}\mu : \; s \leq f^+, \; s \text{ simple measurable} \right\}. \tag{1.29}$$

Definition 1.3.3. Let $f : X \to [-\infty, +\infty]$ be an $\mathcal{M}$-measurable function. Its *integral* $\int f \mathrm{d}\mu$ is

$$\int f \, \mathrm{d}\mu := \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu \tag{1.30}$$

provided that $\int f^+ \mathrm{d}\mu < \infty$ *or* $\int f^- \mathrm{d}\mu < \infty$: we want to avoid a situation $\infty - \infty$ here. If $\int f^+ \mathrm{d}\mu < \infty$ *and* $\int f^- \mathrm{d}\mu < \infty$ then f is called *μ-integrable*. Let $f : X \to \mathbb{C}$ be $\mathcal{M}$-measurable such that $|f| : X \to \mathbb{R}$ is μ-integrable. The μ-integral of f is defined by

$$\int f \, \mathrm{d}\mu := \int \Re f \, \mathrm{d}\mu + \mathrm{i} \int \Im f \, \mathrm{d}\mu, \tag{1.31}$$

where $\Re f, \Im f : X \to \mathbb{R}$ are the real and imaginary parts of f, respectively. If we want to emphasize the variable in the integration, we may write

$$\int f \, \mathrm{d}\mu = \int f(x) \, \mathrm{d}\mu(x),$$

or even $\int f(x)\ \mathrm{d}x$, if the measure is clear from the context. We shall also use the abbreviation

$$\int_E f\ \mathrm{d}\mu := \int \chi_E f\ \mathrm{d}\mu,$$

where $E \in \mathcal{M}$; this is the *integral of f over $E \subset X$.*

1.3.1 Integrating simple functions

It is simple to integrate simple functions. We leave the details as an exercise for the reader:

Exercise 1.3.4. Let $r, s : X \to [0, \infty[$ be $\mathcal{M}$-measurable simple functions and $a \in [0, \infty[$. Show that

$$\int ar\ \mathrm{d}\mu = a \int r\ \mathrm{d}\mu \quad \text{and} \quad \int (r+s)\ \mathrm{d}\mu = \int r\ \mathrm{d}\mu + \int s\ \mathrm{d}\mu.$$

Moreover, if $r \leq s$, show that $\displaystyle\int r\ \mathrm{d}\mu \leq \int s\ \mathrm{d}\mu$.

1.3.2 Integrating non-negative functions

Let us now concentrate on integrating measurable non-negative functions. As an easy consequence of Exercise 1.3.4, for $\mathcal{M}$-measurable functions $f^+, g^+ : X \to [0, \infty]$ and $a \in \mathbb{R}^+$,

$$\int af^+\ \mathrm{d}\mu = a \int f^+\ \mathrm{d}\mu, \tag{1.32}$$

$$\text{if} \quad f^+ \leq g^+ \quad \text{then} \quad \int f^+\ \mathrm{d}\mu \leq \int g^+\ \mathrm{d}\mu. \tag{1.33}$$

These observations will be used frequently. However, it is not evident whether

$$\int (f^+ + g^+)\ \mathrm{d}\mu = \int f^+\ \mathrm{d}\mu + \int g^+\ \mathrm{d}\mu.$$

This will be obtained as a consequence of the following fundamental result:

Theorem 1.3.5. (Monotone Convergence Theorem.) *For each $k \geq 1$, let $f_k : X \to [0, \infty]$ be $\mathcal{M}$-measurable such that $f_k \leq f_{k+1}$. Then*

$$\lim_{k\to\infty} \int f_k\ \mathrm{d}\mu = \int \lim_{k\to\infty} f_k\ \mathrm{d}\mu.$$

Proof. Function $f := \lim_{k\to\infty} f_k : X :\to [0,\infty]$ is measurable as a limit of measurable functions. Clearly, $f_k \leq f_{k+1} \leq f$, so the increasing sequence of integrals $\int f_k \mathrm{d}\mu \leq \int f \mathrm{d}\mu$ converges to the limit

$$\lim_{k\to\infty} \int f_k \; \mathrm{d}\mu \leq \int f \; \mathrm{d}\mu.$$

Let $0 < \varepsilon < 1$. Take a simple measurable function s such that $s \leq f$ and

$$\int s \; \mathrm{d}\mu \geq (1-\varepsilon) \int f \; \mathrm{d}\mu.$$

Let $E_k := \{f_k > (1-\varepsilon)s\}$. Since f_k and s are measurable, $E_k \in \mathcal{M}$. Furthermore,

$$\begin{aligned}
\int f_k \; \mathrm{d}\mu &\geq \int (1-\varepsilon) s \; \chi_{E_k} \; \mathrm{d}\mu \\
&= \sum_{a\in s(X)} (1-\varepsilon) a \cdot \mu\left(E_k \cap \{s = a\}\right) \\
&\xrightarrow[k\to\infty]{} (1-\varepsilon) \sum_{a\in s(X)} a \cdot \mu\left(\{s = a\}\right) \\
&= (1-\varepsilon) \int s \; \mathrm{d}\mu \\
&\geq (1-\varepsilon)^2 \int f \; \mathrm{d}\mu,
\end{aligned}$$

where the limit is due to $X = \bigcup_{k=1}^{\infty} E_k$, where $E_k \subset E_{k+1} \in \mathcal{M}$. Thus

$$\lim_{k\to\infty} \int f_k \; \mathrm{d}\mu \geq (1-\varepsilon)^2 \int f \; \mathrm{d}\mu.$$

Taking $\varepsilon \to 0$, the proof is complete. □

Corollary 1.3.6. *Let $f, g : X \to [0,\infty]$ be $\mathcal{M}$-measurable. Then*

$$\int (f+g) \; \mathrm{d}\mu = \int f \; \mathrm{d}\mu + \int g \; \mathrm{d}\mu.$$

Proof. Take measurable simple functions $f_k, g_k : X \to [0,\infty[$ such that $f_k \leq f_{k+1}$ and $g_k \leq g_{k+1}$ for each $k \in \mathbb{Z}^+$, and $f_k \to f$ and $g_k \to g$ pointwise. Then $f_k + g_k : X \to [0,\infty[$ is simple measurable such that

$$f_k + g_k \leq f_{k+1} + g_{k+1} \xrightarrow[k\to\infty]{} f + g,$$

so that by the Monotone Convergence Theorem 1.3.5,

$$\begin{aligned}\int (f+g)\ \mathrm{d}\mu &= \lim_{k\to\infty}\int (f_k+g_k)\ \mathrm{d}\mu \\ &\overset{\text{Exercise 1.3.4}}{=} \lim_{k\to\infty}\left(\int f_k\ \mathrm{d}\mu + \int g_k\ \mathrm{d}\mu\right) \\ &= \int f\ \mathrm{d}\mu + \int g\ \mathrm{d}\mu,\end{aligned}$$

establishing the result. □

Corollary 1.3.7. *Let $g_j : X \to [0,\infty]$ be $\mathcal{M}$-measurable for each $j \in \mathbb{Z}^+$. Then*

$$\int \sum_{j=1}^{\infty} g_j\ \mathrm{d}\mu = \sum_{j=1}^{\infty}\int g_j\ \mathrm{d}\mu.$$

Proof. For each $k \in \mathbb{Z}^+$, let us define functions $f_k, f : X \to [0,\infty]$ by

$$f_k := \sum_{j=1}^{k} g_j \quad \text{and} \quad f := \lim_{k\to\infty} f_k = \sum_{j=1}^{\infty} g_j.$$

These functions are measurable and $f_k \le f_{k+1} \le f$, so

$$\begin{aligned}\int \lim_{k\to\infty}\sum_{j=1}^{k} g_j\ \mathrm{d}\mu &\overset{\text{Monotone Convergence}}{=} \lim_{k\to\infty}\int\sum_{j=1}^{k} g_j\ \mathrm{d}\mu \\ &\overset{\text{Corollary 1.3.6}}{=} \lim_{k\to\infty}\sum_{j=1}^{k}\int g_j\ \mathrm{d}\mu,\end{aligned}$$

completing the proof. □

Theorem 1.3.8. (Fatou's Lemma.) *Let $g_k : X \to [0,\infty]$ be $\mathcal{M}$-measurable for each $k \in \mathbb{Z}^+$. Then*

$$\int \liminf_{k\to\infty} g_k\ \mathrm{d}\mu \le \liminf_{k\to\infty}\int g_k\ \mathrm{d}\mu.$$

Proof. Notice that

$$\liminf_{k\to\infty} g_k = \sup_{k\ge 1}\inf_{j\ge k} g_j.$$

Define $f_k := \inf_{j \geq k} g_j$ for each $k \geq 1$. Now $f_k : X \to [0, \infty]$ is measurable and $f_k \leq f_{k+1}$, so that $\sup_{k\geq 1} f_k = \lim_{k\to\infty} f_k$, and

$$\begin{aligned}
\int \liminf_{k\to\infty} g_k \, \mathrm{d}\mu &= \int \sup_{k\geq 1} f_k \, \mathrm{d}\mu \\
&= \int \lim_{k\to\infty} f_k \, \mathrm{d}\mu \\
&\overset{\text{Monotone Convergence}}{=} \lim_{k\to\infty} \int f_k \, \mathrm{d}\mu \\
&= \liminf_{k\to\infty} \int f_k \, \mathrm{d}\mu \\
&\leq \liminf_{k\to\infty} \int g_k \, \mathrm{d}\mu.
\end{aligned}$$

The proof is complete. □

Exercise 1.3.9. Sometimes $\int \liminf_{k\to\infty} g_k \, \mathrm{d}\mu < \liminf_{k\to\infty} \int g_k \, \mathrm{d}\mu$ happens in Fatou's Lemma 1.3.8. Find an example.

Exercise 1.3.10. Actually, the Monotone Convergence Theorem 1.3.5 and Fatou's Lemma 1.3.8 are logically equivalent: prove this.

1.3.3 Integration in general

Let $f : X \to [-\infty, +\infty]$ be an $\mathcal{M}$-measurable function. Recall that if

$$I^+ = \int f^+ \, \mathrm{d}\mu < \infty \qquad \textit{or} \qquad I^- = \int f^- \, \mathrm{d}\mu < \infty$$

then the μ-integral f is $\int f \mathrm{d}\mu = I^+ - I^-$. Moreover, if both I^+ *and* I^- are finite, f is called μ-integrable. We shall be interested mainly in μ-integrable functions.

Theorem 1.3.11. *Let $a \in \mathbb{R}$ and $f : X \to [-\infty, +\infty]$ be μ-integrable. Then*

$$\int af \, \mathrm{d}\mu = a \int f \, \mathrm{d}\mu. \tag{1.34}$$

Moreover, if $g : X \to [-\infty, +\infty]$ is μ-integrable such that $f \leq g$, then

$$\int f \, \mathrm{d}\mu \leq \int g \, \mathrm{d}\mu. \tag{1.35}$$

Especially, $\left|\int f \, \mathrm{d}\mu\right| \leq \int |f| \, \mathrm{d}\mu.$

Exercise 1.3.12. Prove Theorem 1.3.11.

Exercise 1.3.13. Let $E \in \mathcal{M}$ and $|f| \leq g$, where f is $\mathcal{M}$-measurable and g is μ-integrable. Show that f and $f\chi_E$ are μ-integrable.

We continue with noticing the short-sightedness of integrals:

Lemma 1.3.14. *Let $f, g : X \to [-\infty, +\infty]$ be μ-integrable. Then*

1. *Let $E \in \mathcal{M}$ such that $\mu(E) = 0$. Then $\int_E f \cdot \mathrm{d}\mu = 0$.*
2. *Let $f = g$ μ-almost everywhere. Then $\int f \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu$.*
3. *Let $\int |f| \, \mathrm{d}\mu = 0$. Then $f = 0$ μ-almost everywhere.*

Proof. First,

$$\begin{aligned} \int_E f^+ \, \mathrm{d}\mu &= \int f^+ \chi_E \, \mathrm{d}\mu \\ &= \sup \left\{ \int s \, \mathrm{d}\mu : \ s \leq f^+ \chi_E \text{ simple measurable} \right\} \\ &\overset{\mu(E)=0}{=} 0, \end{aligned}$$

proving the first result. Next, let us suppose $f = g$ μ-almost everywhere. Then

$$\begin{aligned} \int f^+ \, \mathrm{d}\mu &= \int \left(f^+ \chi_{\{f=g\}} + f^+ \chi_{\{f \neq g\}} \right) \, \mathrm{d}\mu \\ &\overset{\text{Corollary 1.3.6}}{=} \int_{\{f=g\}} f^+ \, \mathrm{d}\mu + \int_{\{f \neq g\}} f^+ \, \mathrm{d}\mu \\ &\overset{\mu(\{f \neq g\})=0}{=} \int_{\{f=g\}} f^+ \, \mathrm{d}\mu, \end{aligned}$$

showing that $\int f^+ \mathrm{d}\mu = \int g^+ \mathrm{d}\mu$, establishing the second result. Finally,

$$
\begin{aligned}
\mu(\{f \neq 0\}) &= \mu\Big(\bigcup_{k=1}^{\infty} \{|f| > 1/k\}\Big) \\
&\leq \sum_{k=1}^{\infty} \mu(\{|f| > 1/k\}) \\
&= \sum_{k=1}^{\infty} \int \chi_{\{|f|>1/k\}} \, \mathrm{d}\mu \\
&\leq \sum_{k=1}^{\infty} \int k|f| \, \mathrm{d}\mu \\
&= \sum_{k=1}^{\infty} k \int |f| \, \mathrm{d}\mu
\end{aligned}
$$

so that if $\int |f| \mathrm{d}\mu = 0$, then $\mu(\{f \neq 0\}) = 0$. □

Proposition 1.3.15. *Let $f : X \to [-\infty, +\infty]$ be μ-integrable. Then $f(x) \in \mathbb{R}$ for μ-almost every $x \in X$.*

Proof. First, $\{f^+ = \infty\} = \bigcap_{k=1}^{\infty} \{f^+ > k\} \in \mathcal{M}$, because f^+ is $\mathcal{M}$-measurable. Thereby

$$
\begin{aligned}
\mu(\{f^+ = \infty\}) &= \frac{1}{k} \int k \cdot \chi_{\{f^+=\infty\}} \, \mathrm{d}\mu \\
&\leq \frac{1}{k} \int f^+ \, \mathrm{d}\mu \quad \xrightarrow[k\to\infty]{} \quad 0,
\end{aligned}
$$

so that $\mu(\{f^+ = \infty\}) = 0$. Similarly, $\mu(\{f^- = \infty\}) = 0$. □

Remark 1.3.16. By Lemma 1.3.14 and Proposition 1.3.15, when it comes to integration, we may identify a μ-integrable function $f : X \to [-\infty, +\infty]$ with the function $\tilde{f} : X \to \mathbb{R}$ defined by

$$
\tilde{f}(x) = \begin{cases} f(x), & \text{when } f(x) \in \mathbb{R}, \\ 0, & \text{when } |f(x)| = \infty. \end{cases}
$$

We shall do this identification without a further notice.

Theorem 1.3.17. *Let $f, g : X \to [-\infty, +\infty]$ be μ-integrable. Then $f + g$ is μ-integrable and*

$$\int (f+g)\ \mathrm{d}\mu = \int f\ \mathrm{d}\mu + \int g\ \mathrm{d}\mu. \tag{1.36}$$

Proof. For integrable $f, g : X \to \mathbb{R}$, the function $f + g : X \to \mathbb{R}$ is measurable. Notice that

$$f + g = \begin{cases} (f^+ - f^-) + (g^+ - g^-) \\ (f+g)^+ - (f+g)^-. \end{cases}$$

Since $(f+g)^+ \leq f^+ + g^+$, and $(f+g)^- \leq f^- + g^-$, the integrability of $f + g$ follows. Moreover, $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$. By Corollary 1.3.6,

$$\int (f+g)^+ \mathrm{d}\mu + \int f^- \mathrm{d}\mu + \int g^- \mathrm{d}\mu = \int (f+g)^- \mathrm{d}\mu + \int f^+ \mathrm{d}\mu + \int g^+ \mathrm{d}\mu,$$

implying

$$\begin{aligned} \int (f+g)\ \mathrm{d}\mu &= \int (f+g)^+\ \mathrm{d}\mu - \int (f+g)^-\ \mathrm{d}\mu \\ &= \int f^+\ \mathrm{d}\mu - \int f^-\ \mathrm{d}\mu + \int g^+\ \mathrm{d}\mu - \int g^-\ \mathrm{d}\mu \\ &= \int f\ \mathrm{d}\mu + \int g\ \mathrm{d}\mu. \end{aligned}$$

The proof for the summation is thus complete. □

Theorem 1.3.18. (Lebesgue's Dominated Convergence Theorem.) *For each $k \geq 1$, let $f_k : X \to [-\infty, +\infty]$ be measurable and $f_k \xrightarrow[k\to\infty]{} f$. Assume that $|f_k| \leq g$ for every $k \geq 1$, where g is μ-integrable. Then*

$$\begin{aligned} \int |f_k - f|\ \mathrm{d}\mu &\xrightarrow[k\to\infty]{} 0, \\ \int f_k\ \mathrm{d}\mu &\xrightarrow[k\to\infty]{} \int f\ \mathrm{d}\mu. \end{aligned}$$

Proof. Functions $f_k, f, |f_k - f|$ are μ-integrable, because they are measurable, g is μ-integrable, $|f_k|, |f| \leq g$ and $|f_k - f| \leq 2g$. For each $k \geq 1$, define function $g_k := 2g - |f_k - f|$. Then functions $g_k \geq 0$ satisfy the assumptions of Fatou's Lemma 1.3.8, yielding

$$\begin{aligned}\int 2g \; d\mu &= \int \liminf_{k\to\infty} g_k \; d\mu \\ &\overset{\text{Fatou}}{\leq} \liminf_{k\to\infty} \int g_k \; d\mu \\ &= \liminf_{k\to\infty} \left(\int 2g \; d\mu - \int |f_k - f| \; d\mu \right) \\ &= \int 2g \; d\mu - \limsup_{k\to\infty} \int |f_k - f| \; d\mu.\end{aligned}$$

Here we may cancel $\int 2g \; d\mu \in \mathbb{R}$, getting

$$\limsup_{k\to\infty} \int |f_k - f| \; d\mu \leq 0,$$

so that $\int |f_k - f| \; d\mu \xrightarrow[k\to\infty]{} 0$. Finally,

$$\left| \int f_k \; d\mu - \int f \; d\mu \right| = \left| \int (f_k - f) \; d\mu \right| \leq \int |f_k - f| \; d\mu \xrightarrow[k\to\infty]{} 0,$$

which completes the proof. □

Proposition 1.3.19. *Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann-integrable on each closed interval $[a,b] \subset \mathbb{R}$. Then $f\chi_{[a,b]}$ is Lebesgue-integrable and the Riemann- and Lebesgue-integrals coincide:*

$$\int_a^b f(x) \; dx = \int_{[a,b]} f \; d\lambda_{\mathbb{R}}.$$

Exercise 1.3.20. Prove Proposition 1.3.19. Recall the definition of the Riemann-integral: Let $g : [a,b] \to \mathbb{R}$ be bounded. A finite sequence $P_n = (x_0, \cdots, x_n)$ is called *partition of* $[a,b]$ if

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

for which the *lower and upper Riemann sums* $L(g, P_n), U(g, P_n)$ are defined by

$$\begin{aligned} U(g, P_n) &= \sum_{k=1}^{n} \left(\sup_{x_{k-1} \leq x < x_k} g(x) \right) (x_k - x_{k-1}), \\ L(g, P_n) &= \sum_{k=1}^{n} \left(\inf_{x_{k-1} \leq x < x_k} g(x) \right) (x_k - x_{k-1}). \end{aligned}$$

Now $L(g) \leq U(g)$, where

$$\begin{cases} U(g) := \inf \{U(g,P) : \; P \text{ is a partition of } [a,b]\}, \\ L(g) := \sup \{L(g,P) : \; P \text{ is a partition of } [a,b]\}. \end{cases}$$

If $L(g) = U(g)$, we say that g is *Riemann-integrable* with Riemann-integral

$$\int_a^b g(x) \; \mathrm{d}x = L(g).$$