Mat-1.3608 Markov chains. We use the notation from Nummelin's paper. The chain moves according to the kernel $P(x, A)$, where

$$P(x, A) := \mathbb{P}(X_n \in A | X_{n-1} = x) = \int_A p(x, y) dy + r(x)\delta_x(A),$$

where $r(x)$ is the probability $r(x) = \mathbb{P}(X_{n+1} = x | X_n = x)$.

III Exercise 7.2. 2008 Tikanmäki/Valkeila.

1. Let P be the tarnsition kernel and let λ be the intitial distribution. Write the probability of the event
$$\mathbb{P}(X_0 \in A_0, \ldots, X_n \in A_n)$$
using the kernel P and initial distribution.
2. Let X be a Markov chain, and let $k < n < m$. Then
$$\mathbb{P}(X_k \in A, X_m \in C | X_n \in B) = \mathbb{P}(X_k \in A | X_n \in B)\mathbb{P}(X_m \in C | X_n \in B).$$
["The past and future are independent given the present.“]
3. [Metropolis-Hastings algorithm] Check that the M-H algorithm, described in Nummelin Example 1, is reversible and hence π is its invariant distribution.
4. Let Y_n, $n \geq 1$ be independent identically distributed real random variables, and let $\mathbb{P}(Y_n \in A) = \int_A g(z)dz$. Let X_0 be a random variable, independent on Y_n, $n \geq 1$, and $\mathbb{P}(X_0 \in A) = \int_A \lambda(x)dx$. Show that *random walk*
$$X_n = X_0 + \sum_{k=1}^{n} Y_k$$
is a Markov chain, describe the transition kernel P, and the densities $p(x, y)$ and $r(x)$.
5. [Continuation] Assume that $X_0 \geq 0$ and define the *reflected random walk* by
$$W_0 = X_0 \quad \text{and} \quad W_n = \max(W_{n-1} + Y_n, 0).$$
Show that W is a Markov chain and find its P.
6. [Application of the strong Markov property] Let X be an irreducible Markov chain in finite state space S. Assume that we observe the chain only when it moves. More formally, let $\tau_0 = 0$ and τ_{m+1} is given by
$$\tau_{m+1} = \inf \{n \geq \tau_m | X_n \neq X_{\tau_m}\},$$
and put $Z_m = X_{\tau_m}$. Show that Z is a Markov chain with the transition marix $\tilde{p}_{ij} = \frac{p_{ij}}{\sum_{k \neq i} p_{ik}}$.