The Geometry of Eigenanalysis

with Applications to Structural Stability

Computational eigenproblem solving methods for non-linear systems can be broadly divided in two categories: direct methods and approximative methods. After briefly recalling those classical methods of linear algebra, we shall make an attempt to interpret geometrically the algebraic objects involved. Whenever the eigenvector solution can be interpreted as either the primary tangent direction or the branching tangent direction to the equilibrium set at a singular point, then it makes sense to asses the sensitivity of the eigenvector solution with respect to the entries of the singular jacobian matrix. As we shall see, the sensitivity of the eigenvector depends essentially on the geometry of the singular matrix submanifold embedded in the ambient matrix space. Therefore by investigating the neighbourhood - on the submanifold - of a singular jacobian matrix associated to a given physical system, we can actually evaluate *a priori* the error with respect to the eigenvector.

For an engineer doing structural stability analysis, the eigenvector represents a buckling mode for a geometrically perfect system. The investigation of initial imperfection sensitivity therefore relies on a correct assumption of the previously computed buckling mode. However, in engineering applications eigenvectors to non-linear eigenproblems are solved very often in an approximative way, which places the a-priori eigenvector error analysis to an incontournable position.

Computational eigenanalysis by direct methods

By direct methods we designate computational methods which use a variant of the Newton's method to solve the system of equations build up from the equilibrium equations augmented by the criticality condition and a well chosen normalization condition for the eigenvector. In the particular case of a limit point, such an augmented system of equations is often referred to in the literature as the Moore-Spence system [7, 2], although the Keener-Keller method [3] often pops up as well. The study of numerical methods for solving eigenproblems related to more complex bifurcation points has also received due attention [4, 5, 10, 6], even though there are still many pending questions.

Note that for an eigenproblem associated to a limit point, the initial guess can be obtained by following numerically the primary equilibrium path close enough to the limit point: the initial guess then consists of the concatenation of a regular point on the equilibrium path and a tangent vector at that point. As far as symmetric pitchfork bifurcation points are concerned, the eigenvector associated to the criticality condition can be interpreted geometrically as the generator of the branching direction. Since it is impossible to predict numerically the branching direction from a regular point on the primary equilibrium branch we have no other choice than to take an initial guess consisting of a regular point on the primary branch and a *random* eigenvector. As a corollary we can not fully control the convergence of Newton's method, and it is one of

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the reasons why direct methods have not been yet implemented in commercial finite element packages.

Computational eigenanalysis by approximative methods

In contrast to direct methods, approximative methods do not inherently intend to give exact solutions to non-linear eigenproblems with arbitrary precision. Approximative methods are based on truncated Taylor series of the equilibrium and criticality conditions with respect to the critical parameter and evaluated at the reference state. Polynomial eigenproblems can always be reformulated as generalized linear eigenproblems for which robust solvers are available. This is a reason why in commercial finite element packages the standard solution for non-linear eigenproblems is the linearisation, i.e. Taylor series of the eigenproblem truncated after the first term. Although linearised eigenproblem solvers are robust and always give real valued solutions for conservative systems, it is often hard to interpret the meaning of those solutions with respect to the original non-linear system.

Geometric interpretation of equilibrium states

Equilibrium states of parameter dependent dynamical systems can be defined as the set of all steady state solutions for authorized external parameter values. If we restrict ourselves to finite dimensional systems, either inherently or due to a discretization process, the equilibrium set is then the zero set of a smooth vector valued map defined on the configuration manifold. Note that if the equilibrium defining map is a polynomial, then the equilibrium set is an algebraic variety. On the other hand, if the equilibrium defining map is a submersion, then the equilibrium set is a smooth manifold. Investigation of neighbourhoods of regular points on the equilibrium set can therefore be conducted using standard smooth manifold theory, but if we want to investigate neighbourhoods of singular points in full generality and extent using tools from algebraic geometry we have to restrict our equilibrium defining maps to polynomial ones.

Singularity theory [8, 9, 1] trades some of the generality by restricting the investigation to transversal singularities, which then enables to develop results similar to algebraic geometry for smooth equilibrium defining maps. Classical approaches in singularity theory heavily rely on a technique called the Lyapunov-Schmidt reduction, which enables to investigate some properties of singular points in multi-dimensional or Banach spaces by investigating the corresponding properties in reduced - two dimensional - spaces. It can be shown that in the neighbourhood of a singular point each individual equilibrium branch is diffeomorphic to it's counterpart in the reduced space, and additionally that transversality is preserved by the Lyapunov-Schmidt reduction. This important proposition enables us to locally *identify* the equilibrium set in the state space with its image in the reduced space. In particular tangent cones in the state space and in the reduced space can be identified.

Although it is possible to develop computational algorithms based on singularity theory such that they directly compute the tangent cone at a singular equilibrium point, that sort of approach necessitates the computation of second order derivatives of the equilibrium defining map. Is it possible to get some information about the tangent cone without the computation of second order derivatives? It turns out that yes, it is in some special cases. First note that in the case of a limit point, which is not a singular point in the geometric sense, the tangent space to the primary branch is equal to the kernel of the jacobian matrix at that point. It can be also shown that if two equilibrium branches intersect *orthogonally* in the state space, then the tangent space to the secondary branch is equal to the kernel of the jacobian matrix. In particular the case of symmetric pitchfork bifurcation falls into this category, where the branching direction can be identified with the eigenvector to the singular jacobian matrix.

A-priori eigenvector error analysis

Assuming that the branching direction can be identified with the eigenvector and that the eigenproblem is solved in an approximate way, it is of great interest to know a-priori, i.e. without computing the exact solution, how much the approximately found branching direction differs form the exact one. It can be shown that the unit length eigenvector can be computed from the entries of the singular jacobian matrix, and therefore that it makes sense to define a *unit eigenvector map* defined on the singular matrix submanifold. We shall then define the *unit eigenvector error map* between two points on the singular matrix submanifold as the angle between the two corresponding unit eigenvectors. Note that this error map would correspond to an a-posteriori error analysis since we need to know both points.

For an a-priori error analysis, we have to choose one center point and compute the error map with respect to a point lying on a *geodesic sphere* of radius ϵ around the center point. One can then compute minimum, maximum and average error values at a distance ϵ form the center point. It can be shown that there are some locations on the singular matrix submanifold that are more error prone than others. In particular, neighbourhoods of points of higher rank deficiency, which are points on the boundary of the singular - rank deficient 1 - matrix submanifolds, constitute risk zones for approximate eigenanalysis.

The particular case of 2×2 real symmetric matrices provide us an illustrative example of how things work in higher dimensions. Figure shows two case of behaviour of the unit eigenvector map defined on the singular matrix submanifold $\mathcal{M}$ of symmetric 2×2 rank 1 matrices. On the left, we have considered the case where the center point $\boldsymbol{A} \in \mathcal{M}$ is far from the set of higher rank deficiency matrices $\mathcal{C}$. The image of the geodesic sphere $\mathcal{S}$ by the unit eigenvector map $\boldsymbol{q}$ is then concentrated around the image point $\boldsymbol{q}(\boldsymbol{A})$. On the right we have considered the case when the center point is close to the set $\mathcal{C}$. In this case we can observe a high deviation of points from the image set $\boldsymbol{q}(\mathcal{S})$ compared to the reference point $\boldsymbol{q}(\boldsymbol{A})$.

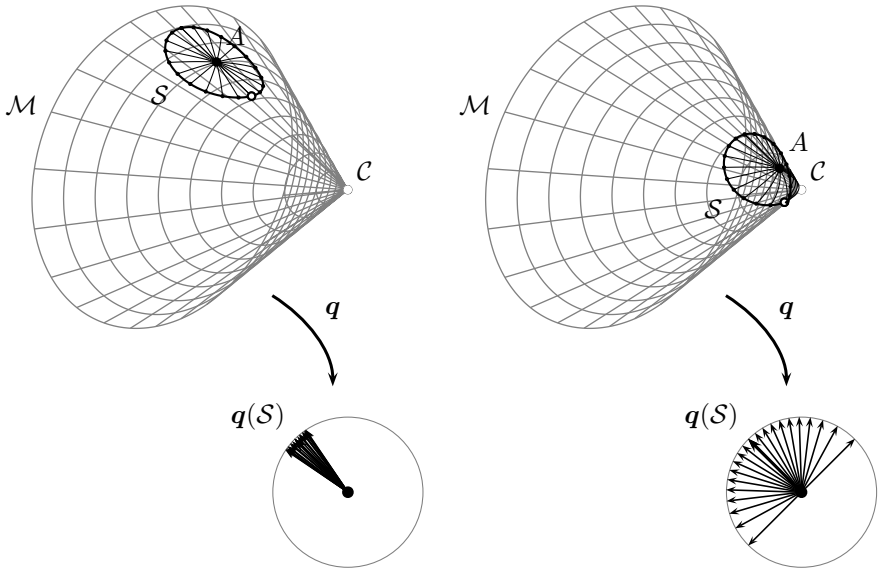

Figure 1: Cases of behaviour of the unit eigenvector map in a small neighborhood

If we consider ϵ the radius of the $\boldsymbol{A}$-centred geodesic sphere $\mathcal{S}$ and ρ the shortest distance from the center point $\boldsymbol{A}$ to the set of higher rank deficiency matrices $\mathcal{C}$, then we may plot maximum, minimum and average errors versus the relative distance ϵ/ρ, as we did in Figure .

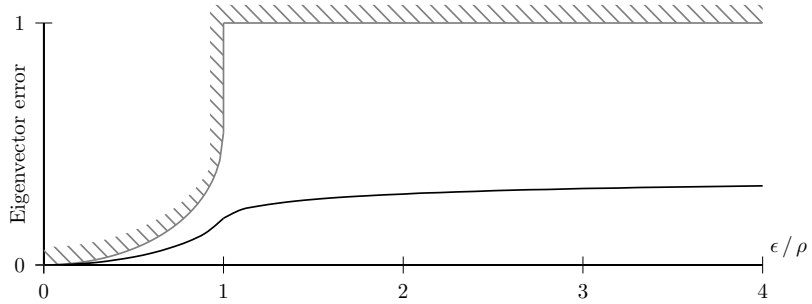

Figure 2: Maximum, minimum and average eigenvector error plot

Although the minimum error is a constant zero function, it is easy to see a qualitative jump in the maximum and average error at the relative distance $\epsilon/\rho = 1$, a result which corroborates with the claims that we have set previously forth.

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