AV8 KP3

1 a) $y''' = -e^t y' - y = g(t, y, y')$

ASETETAAN $x_1 = y$, $x_2 = y'$, $x_3 = y''$

NYT SIIS: $\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ x_3' = y''' = g(t, x_1, x_2, x_3) \end{cases}$

$$\Rightarrow \quad \bar{x}'(t) = \begin{bmatrix} y'(t) \\ y''(t) \\ y'''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -e^t & 0 \end{bmatrix} \bar{x}(t)$$

b) $y'' = -k \sin y$

ASET. $x_1 = y$, $x_2 = y'$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -k \sin x_1 \end{cases}$$

c) $\begin{cases} y_1'' - y_1' - 2y_1 = t^2 & \Rightarrow \quad y_1'' = y_1' + 2y_1 + t^2 \\ y_2'' - y_2 - 3y_1 = 0 & \Rightarrow \quad y_2'' = y_2 + 3y_1 \end{cases}$

ASET. $x_1 = y_1$, $x_2 = y_1'$, $x_3 = y_2$, $x_4 = y_2'$

$$\bar{x}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} y_1' \\ y_1'' \\ y_2' \\ y_2'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ t^2 \\ 0 \\ 0 \end{bmatrix}$$

AV 8

2. $\bar{Y}' = A\bar{Y}$ $\quad \lambda_1 = 1$ $\quad \bar{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$ $\quad \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Y(0) = \begin{bmatrix} -0{,}1 \\ 0{,}1 \end{bmatrix}$$

YLEINEN RATKAISU

$$\bar{Y}(t) = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2$$

$$= c_1 e^{t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Y(0) = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0{,}1 \\ 0{,}1 \end{bmatrix}$$

PERINTEINEN YHTÄLÖPARIRATKAISU.
VOI TOKI GAUSSAILLA ...

$$\begin{cases} -2c_1 + c_2 = -0{,}1 \\ c_1 + c_2 = 0{,}1 \end{cases}$$

$$c_1 = 0{,}1 - c_2$$

$$-0{,}2 + 2c_2 + c_2 = -0{,}1$$

$$3c_2 = 0{,}1$$

$$c_2 = \frac{1}{30}$$

$$c_1 = \frac{2}{30}$$

$$Y(0) = \begin{bmatrix} -0{,}1 \\ 0{,}1 \end{bmatrix} \qquad Y_1 = -0{,}1 \quad Y_2 = 0{,}1$$

$$Y(1) = \frac{2e}{30}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \frac{e^2}{30}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4e + e^2 \\ 2e + e^2 \end{bmatrix} \cdot \frac{1}{30} \Rightarrow Y_1 \approx -0{,}116 \quad Y_2 \approx 0{,}428$$

$$Y(1{,}5) = \frac{2e^{1{,}5}}{30}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \frac{e^3}{30}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4e^{1{,}5} + e^3 \\ 2e^{1{,}5} + e^3 \end{bmatrix} \cdot \frac{1}{30} \Rightarrow Y_1 \approx 0{,}07196 \quad Y_2 \approx 0{,}968$$

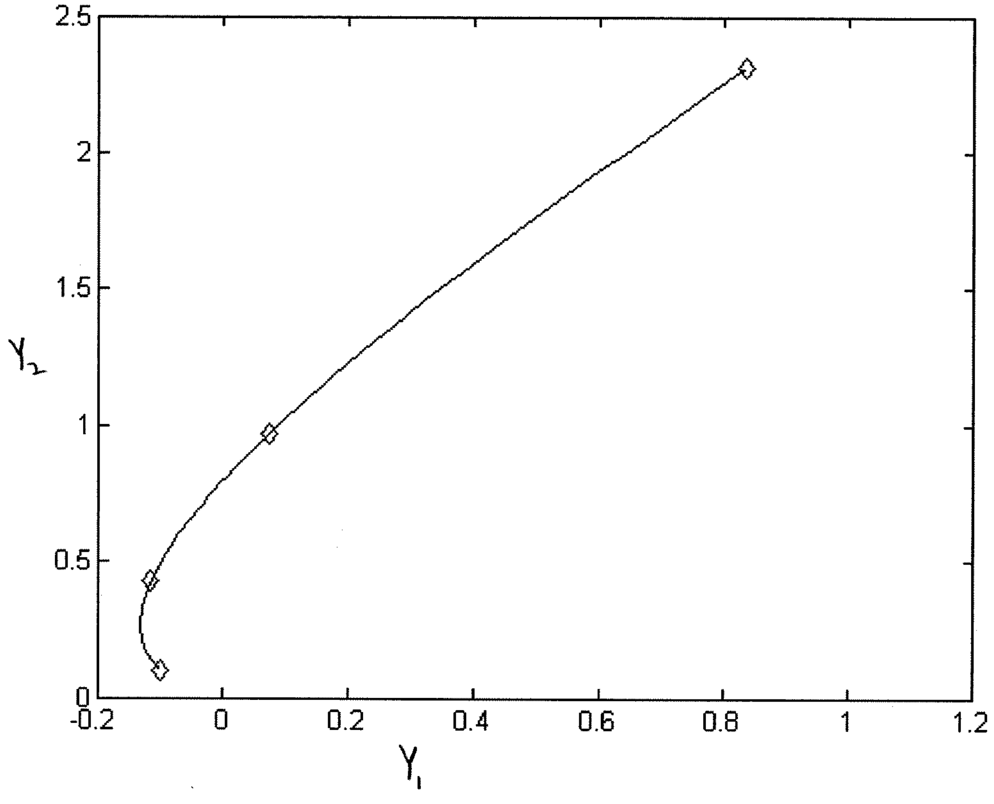
$$Y(2) = \frac{2e^2}{30}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \frac{e^4}{30}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4e^2 + e^4 \\ 2e^2 + e^4 \end{bmatrix} \cdot \frac{1}{30} \Rightarrow Y_1 \approx 0{,}835 \quad Y_2 \approx 2{,}313$$

YL. RATK. SIIS:

$$\bar{Y}(t) = \begin{bmatrix} -4e^{t} + e^{2t} \\ 2e^{t} + e^{2t} \end{bmatrix} \cdot \frac{1}{30}$$

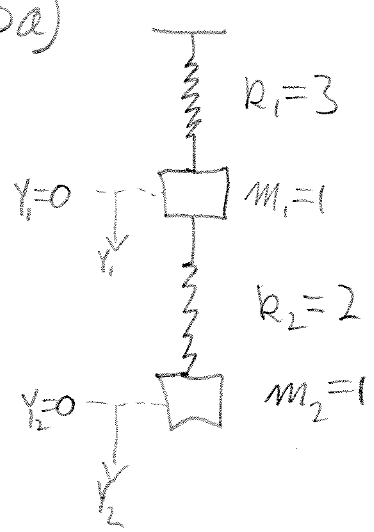
AV8

Tehtävä 2



AV8

3a)



$F = ma \qquad F = -kY$

LIIKEYHTÄLÖT:

$$\begin{cases} m_1 Y_1'' = -k_1 Y_1 + k_2(Y_2 - Y_1) \\ m_2 Y_2'' = -k_2(Y_2 - Y_1) \end{cases}$$

$$\Rightarrow \begin{cases} Y_1'' = -3Y_1 + 2(Y_2 - Y_1) = -5Y_1 + 2Y_2 \\ Y_2'' = -2 \cdot (Y_2 - Y_1) \quad = 2Y_1 - 2Y_2 \end{cases}$$

TOISEN KERTALUVUN SYSTEEMI:

$$\bar{Y}'' = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \bar{Y} \qquad \left(\bar{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \right)$$

1. KL SYYSTEEMIÄ VARTEN ASETETAAN:

$$x_1 = Y_1, \; x_2 = Y_1', \; x_3 = Y_2, \; x_4 = Y_2'$$

NYT SAADAAN RYHMÄ

$$\begin{cases} x_1' = Y_1' = x_2 \\ x_2' = Y_1'' = -5Y_1 + 2Y_2 = -5x_1 + 2x_3 \\ x_3' = Y_2' = x_4 \\ x_4' = Y_2'' = 2Y_1 - 2Y_2 = 2x_1 - 2x_3 \end{cases}$$

MATRIISI MUODOSSA:

$$\bar{X}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \bar{X}, \qquad \bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

AV 8

3b) YRITE $\bar{Y} = e^{\omega t}\bar{X}$

$\bar{Y}'' = A\bar{Y}$, $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$\Rightarrow \omega^2 e^{\omega t}\bar{X} = A(e^{\omega t}\bar{X})$

$A\bar{X} = \omega^2 \bar{X}$

MERKITÄÄN $\omega^2 = \lambda$

OMINAISARVOT:

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (5+\lambda)(2+\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0$$

$\Rightarrow \lambda_1 = -1$

$\lambda_2 = -6$

OMINAISVEKTORIT:

$\lambda_1 = -1$: $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \Rightarrow 2x_1 - x_2 = 0$

VALITAAN $x_1 = 1 \Rightarrow x_2 = 2$

$\Rightarrow \bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = -6$: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 + 2x_2 = 0$

VALITAAN $x_2 = -1 \Rightarrow x_1 = 2$

$\Rightarrow \bar{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

KOSKA $\omega^2 = \lambda$ SAADAAN:

$\omega^2 = -1 \Rightarrow \omega = \pm\sqrt{-1} = \pm i \Rightarrow \omega_1 = i$, $\omega_2 = -i$

$\omega^2 = -6 \Rightarrow \omega = \pm\sqrt{-6} = \pm i\sqrt{6} \Rightarrow \omega_3 = i\sqrt{6}$, $\omega_4 = -i\sqrt{6}$

YLEINEN RATKAISU LINEAARIKOMBINAATIONA:

$$\bar{Y}(t) = C_1 e^{\omega_1 t}\bar{v}_1 + C_2 e^{\omega_2 t}\bar{v}_1 + C_3 e^{\omega_3 t}\bar{v}_2 + C_4 e^{\omega_4 t}\bar{v}_2$$

$$= (C_1 e^{it} + C_2 e^{-it})\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (C_3 e^{i\sqrt{6}t} + C_4 e^{-i\sqrt{6}t})\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(MYÖS) $\left(= (A\cos t + B\sin t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (C\cos(\sqrt{6}t) + D\sin(\sqrt{6}t))\begin{bmatrix} 2 \\ -1 \end{bmatrix} \right)$

$A = C_1 + C_2$, $B = i(C_1 - C_2)$, ...

AV 8

4) $y'' + 2y' + 5y = 0$

$y'' = -2y' - 5y$

$x_1 = y \;,\; x_2 = y'$

$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -5x_1 - 2x_2 \end{cases}$

$\bar{x}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \bar{x}$

(1) $\bar{x}' = A\bar{x}$

YRITE $\bar{x} = \bar{v} e^{\lambda t}$

$\bar{x}' = \lambda \bar{v} e^{\lambda t}$

SIJOITUS (1):SEEN

$\lambda \bar{v} e^{\lambda t} = A \bar{v} e^{\lambda t} \quad | : e^{\lambda t}$

$\lambda \bar{v} = A \bar{v}$

O-ARVOT JA O-VEKTORIT:

$(A - \lambda I)\bar{v} = \bar{0}$

$\begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda) + 5 = \lambda^2 + 2\lambda + 5 = 0$

$\dfrac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = -1 \pm 2i$

O-VEKT.

$\lambda_1 = -1-2i \Rightarrow \begin{bmatrix} 1+2i & 1 \\ -5 & -1+2i \end{bmatrix} \cdot(-1-2i) \sim \begin{bmatrix} 1+2i & 1 \\ 5+10i & 5 \end{bmatrix} \cdot(-5)$

$\sim \begin{bmatrix} 1+2i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow (1+2i)x_1 + x_2 = 0$

KIINNITETÄÄN $x_1 = 1$

$\Rightarrow x_2 = -1-2i$

$\Rightarrow v_1 = \begin{bmatrix} 1 & -1-2i \end{bmatrix}^T$

$v_2 = \begin{bmatrix} 1 & -1+2i \end{bmatrix}^T$

JATKUU $\rightarrow$

AV 8

4) $\xrightarrow{\text{JATKUU}}$

YLEINEN RATKAISU LIN. KOMBINAATIONA

$$\bar{X} = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t}$$

$$\bar{x} = c_1 \begin{bmatrix} 1 \\ -1-2i \end{bmatrix} e^{(-1-2i)t} + c_2 \begin{bmatrix} 1 \\ -1+2i \end{bmatrix} e^{(-1+2i)t}$$

TARKISTETAAN, MITEN KÄVI:

$$Y = x_1 = c_{11} e^{(-1-2i)t} + c_{21} e^{(-1+2i)t}$$

$$= c_{11} \cdot (e^{-t}(\cos 2t - i \sin 2t)) + c_{21} \cdot (e^{-t}(\cos 2t + i \sin 2t))$$

$$= e^{-t} \cdot \left[(c_{11} + c_{21}) \cos 2t + i (c_{21} - c_{11}) \sin 2t \right]$$

$$\Rightarrow Y = e^{-t} \left[c_3 \cos 2t + c_4 \sin 2t \right]$$

JEP, SAMA KUIN LV4.2 HOMOG.-OSA

AV8

5) $Y' = AY \qquad A = \begin{bmatrix} 1/3 & 4/3 \\ 8/3 & 5/3 \end{bmatrix}$

$$\bar{Y} = \bar{v} e^{\lambda t}$$
$$\bar{Y}' = \lambda \bar{v} e^{\lambda t}$$
$$\Rightarrow \lambda \bar{v} e^{\lambda t} = A \bar{v} e^{\lambda t} \quad | : e^{\lambda t}$$
$$A\bar{v} = \lambda \bar{v}$$

O-ARVOT

$$(A - \lambda I)\bar{v} = \bar{0} \Rightarrow \begin{vmatrix} 1/3-\lambda & 4/3 \\ 8/3 & 5/3-\lambda \end{vmatrix} = (1/3-\lambda)(5/3-\lambda) - \frac{32}{9} = 0$$
$$= \lambda^2 - 2\lambda - 3 = 0$$
$$\Rightarrow \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2$$
$$\begin{cases} \lambda_1 = 3 \\ \lambda_2 = -1 \end{cases}$$

ORIGO ON SATULA, KOSKA TOINEN O-ARVO ON POSIT. JA TOINEN NEGAT.

O-VEKT.

$$\lambda_1 = 3: \begin{bmatrix} -8/3 & 4/3 \\ 8/3 & -4/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{0}$$
$$\Rightarrow -8/3\, x_1 + 4/3\, x_2 = 0$$
$$-2x_1 + x_2 = 0$$

KIINNIT. $x_1 = 1 \Rightarrow x_2 = 2$

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = -1 : \begin{bmatrix} 4/3 & 4/3 \\ 8/3 & 8/3 \end{bmatrix} \Rightarrow x_1 = -x_2$$
$$\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

YLEINEN RATKAISU:

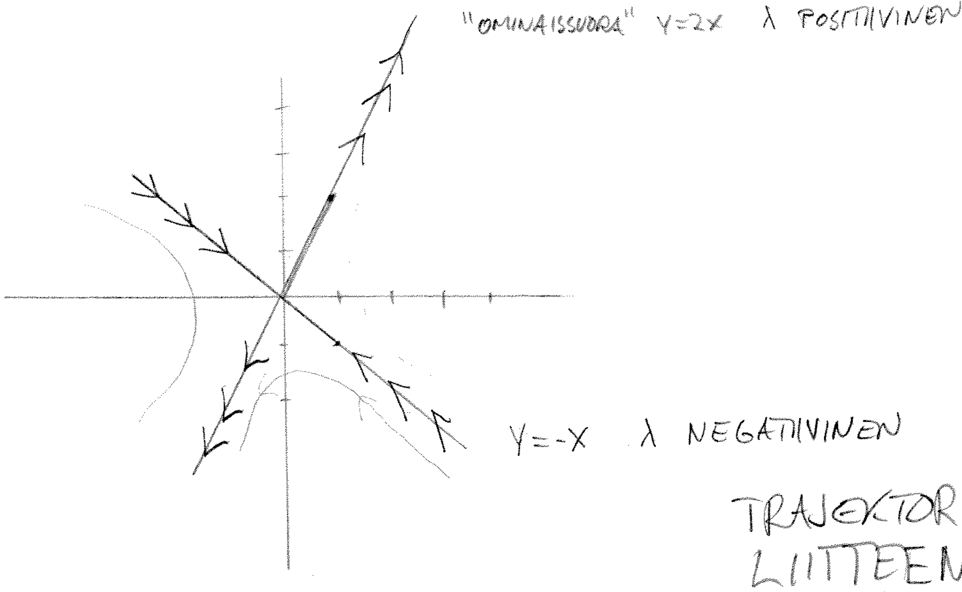
$$\bar{Y} = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

JATKUU →

AV 8

5) $\underset{\rightarrow}{\text{JATKUU}}$

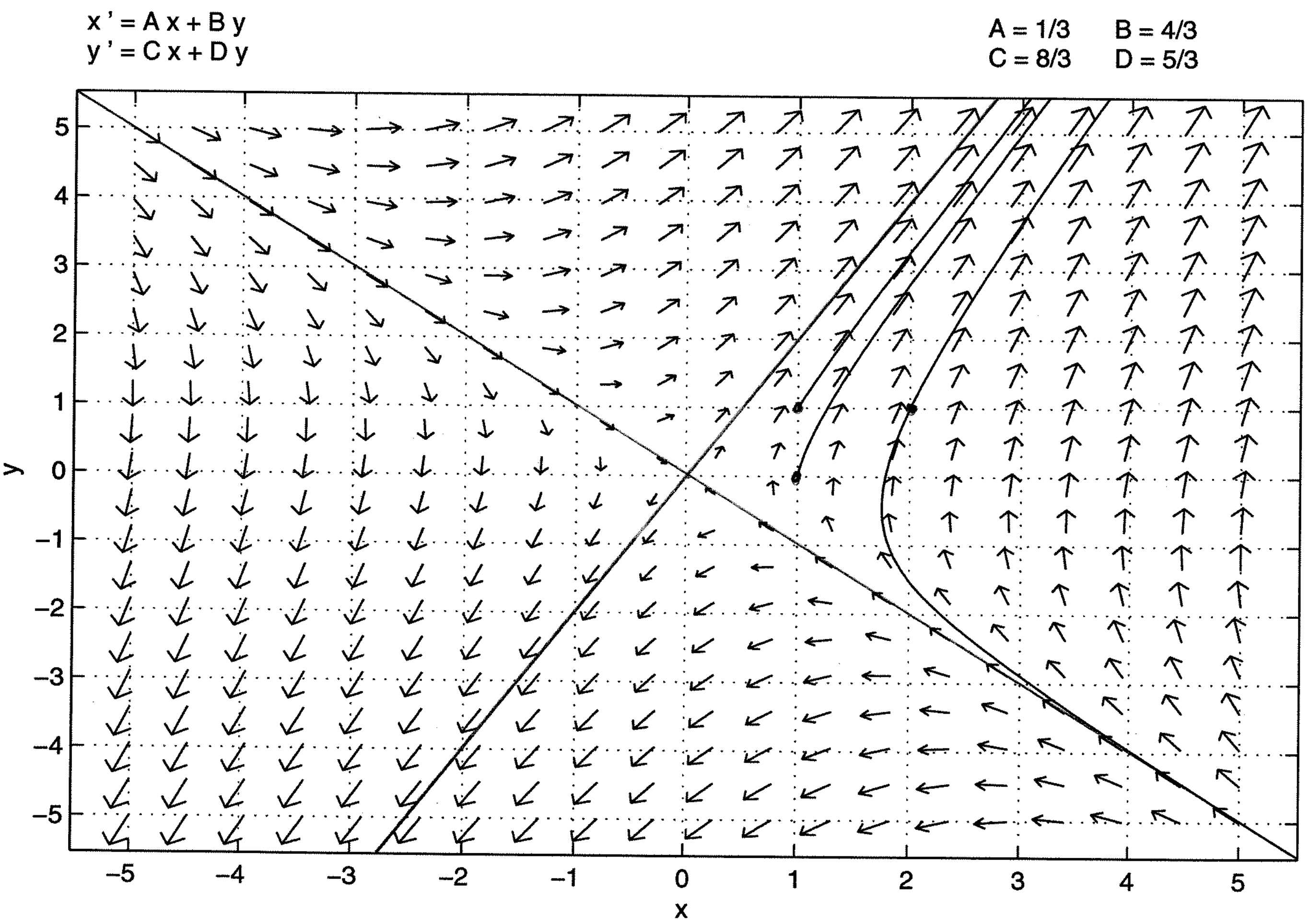
O-VEKTORIT : $\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



TRAJEKTORIHAHMOTELMAT LIITTEENÄ.

(NORMITETTU O-VEKT. KANTA
$\{\bar{v}_{10}, \bar{v}_{20}\} = \{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$)

TEHT 5)



AV 8

6) $\bar{Y}' = A\bar{Y}$ $\bar{Y}(0) = \begin{bmatrix} 0{,}1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} -1/2 & -2 \\ 1 & 3/2 \end{bmatrix}$$

$$\begin{cases} \lambda_1 = \frac{1}{2} + i \\ \lambda_2 = \frac{1}{2} - i \end{cases}$$

$\hookrightarrow$ (EPÄSTABIILI FOKUS ELI) LÄHDESPIRAALI

(REAALIOSA POS.,
"PYÖRTEISYYS" Im-OSASTA)

O-VEKT.

$$\begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \overset{\text{KOMPLEKSIKONJ. KERTOMINEN}}{\sim} \begin{bmatrix} -1-i & -2 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow (-1-i)x_1 = 2x_2$

$x_1 = 1 \Rightarrow x_2 = -\frac{1}{2} - \frac{1}{2}i$

$\Rightarrow \bar{v}_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} - \frac{i}{2} \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} + \frac{i}{2} \end{bmatrix}$

YLEINEN RATKAISU:

$$\bar{Y}(t) = c_1 \bar{v}_1 e^{(\frac{1}{2}+i)t} + c_2 \bar{v}_2 e^{(\frac{1}{2}-i)t} = \begin{bmatrix} e^{(\frac{1}{2}+i)t} & e^{(\frac{1}{2}-i)t} \\ (-\frac{1}{2}-\frac{i}{2})e^{(\frac{1}{2}+i)t} & (-\frac{1}{2}+\frac{i}{2})e^{(\frac{1}{2}-i)t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

ALKUARVOT:

$$\bar{Y}(0) = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2}-\frac{i}{2} & -\frac{1}{2}+\frac{i}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0{,}1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0{,}05 + 0{,}05i \\ 0{,}05 - 0{,}05i \end{bmatrix}$$

$$Y_1 = (0{,}05 + 0{,}05i)e^{(\frac{1}{2}+i)t} + (0{,}05 - 0{,}05i)e^{(\frac{1}{2}-i)t}$$
$$= e^{t/2} \cdot \left[(0{,}05+0{,}05i)e^{it} + (0{,}05-0{,}05i)e^{-it}\right]$$
$$= e^{t/2} \cdot \left[(0{,}05+0{,}05i)\cdot(\cos t + i\sin t) + (0{,}05-0{,}05i)(\cos t - i\sin t)\right]$$
$$= e^{t/2} \cdot \left[0{,}05\cos t + i\,0{,}05\sin t + 0{,}05i\cos t - 0{,}05\sin t \right.$$
$$\left. + 0{,}05\cos t - 0{,}05i\sin t - 0{,}05i\cos t - 0{,}05\sin t\right]$$

JATK. $\rightarrow$

AV8

G) $\overset{\text{JATKUU}}{\longrightarrow}$

$$Y_1 = e^{t/2}\cdot\left[0{,}1\cos t - 0{,}1\sin t\right]$$

$$Y_2 = \left(-\tfrac{1}{2}-\tfrac{i}{2}\right)(0{,}05+0{,}05i)\,e^{(\frac{1}{2}+i)t} + \left(-\tfrac{1}{2}+\tfrac{i}{2}\right)(0{,}05-0{,}05i)\,e^{(\frac{1}{2}-i)t}$$

$$= (-0{,}025-0{,}025i-0{,}025i+0{,}025)\,e^{(\frac{1}{2}+i)t} + 0{,}05i\,e^{(\frac{1}{2}-i)t}$$

$$= e^{t/2}\cdot\left(-0{,}05i\,e^{it} + 0{,}05i\,e^{-it}\right)$$

$$= 0{,}05\cdot e^{t/2}\cdot\left(-i(\cos t + i\sin t) + i(\cos t - i\sin t)\right)$$

$$= 0{,}05\cdot e^{t/2}\,(\sin t - \cancel{i\cos t} + \cancel{i\cos t} + \sin t)$$

$$= 0{,}1\cdot e^{t/2}\sin t$$

AV8

Tehtävä 6