Mat. pk K3/P3 syksy 2005, SLV

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a) $\mathcal{L}^{-1}\left(e^{-as}\frac{1}{s(s-2)}\right) = \mathcal{L}^{-1}\left(e^{-as}\left(\frac{-1/2}{s} + \frac{1/2}{s-2}\right)\right)$ $\boxed{\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)u(t-a)}$

$$= -\frac{1}{2}u(t-a) + \frac{1}{2}e^{2(t-a)}u(t-a)$$

b) $\boxed{(\mathcal{L}f)\cdot(\mathcal{L}g) = \mathcal{L}(f*g)}$; $(f*g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$

$$\mathcal{L}^{-1}\left(e^{-as}\frac{1}{s(s-2)}\right) = \mathcal{L}^{-1}\left(\mathcal{L}(u(t-a))\cdot\mathcal{L}(e^{2t})\right)$$

$$= u(t-a) * e^{2t}$$

$$= \int_0^t u(\tau-a)\,e^{2(t-\tau)}\,d\tau$$

$$= u(t-a)\int_a^t e^{2(t-\tau)}\,d\tau$$

$$= u(t-a)\left[-\frac{1}{2}e^0 + \frac{1}{2}e^{2(t-a)}\right]$$

$$= -\frac{1}{2}u(t-a) + \frac{1}{2}e^{2(t-2)}u(t-a).$$

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a)
$$\begin{cases} y'' + w_0 y = r(t) \\ y(0) = y_0,\quad y'(0) = v_0 \end{cases} \overset{\mathcal{L}}{\Longrightarrow} s^2Y(s) - sy_0 - v_0 + w_0^2Y(s) = R(s)$$

$$\Rightarrow Y(s) = \frac{1}{s^2+w_0^2}\cdot R(s) + y_0\frac{s}{s^2+w_0^2} + v_0\frac{1}{s^2+w_0^2}$$

$$\overset{\mathcal{L}^{-1}}{\Longrightarrow} y(t) = \frac{1}{w_0}\sin(w_0t) * r(t) + y_0\cos(w_0t) + \frac{v_0}{w_0}\sin(w_0t).$$

② b)

$$\sin x \cdot \sin y = \frac{1}{2}\left(-\cos(x+y) + \cos(x-y)\right).$$

$$\sin(\omega_0 t) * \sin(\omega_0 t) = \int_0^t \sin(\omega_0(t-\tau))\sin(\omega_0\tau)\,d\tau$$

$$= \int_0^t \frac{1}{2}\left(-\cos(\omega_0 t) + \cos(\omega_0(t-2\tau))\right) d\tau$$

$$= -\frac{1}{2}\cos(\omega_0 t)\cdot t + \frac{1}{2}\Big|_0^t \frac{1}{-2\omega_0}\sin(\omega_0(t-2\tau))$$

$$= -\frac{1}{2}\cos(\omega_0 t)\cdot t + \frac{1}{2\omega_0}\sin(\omega_0 t)$$

a-kohta

$$\Rightarrow \quad y(t) = \frac{1}{\omega_0}\sin(\omega_0 t) * A\sin(\omega_0 t) + y_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t).$$

$$= \frac{A}{2\omega_0^2}\left(-\omega_0 t\cdot\cos(\omega_0 t) + \sin(\omega_0 t)\right) + \underbrace{y_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)}$$

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③

$$\begin{cases} y_1' = -y_2 + 1 - u(t-1) \\ y_2' = y_1 + 1 - u(t-1) \end{cases}, \quad y_1(0) = y_2(0) = 0$$

$$\overset{\mathcal{L}}{\Rightarrow} \begin{cases} sY_1 = -Y_2 + \frac{1}{s} - \frac{e^{-s}}{s} \\ sY_2 = Y_1 + \frac{1}{s} - \frac{e^{-s}}{s} \end{cases}$$

$$\Rightarrow \begin{cases} Y_1(s) = -\frac{1}{s(s^2+1)} + \frac{e^{-s}}{s(s^2+1)} + \frac{1}{s^2+1} - \frac{e^{-s}}{s^2+1} \\ Y_2(s) = \frac{1}{s^2+1} - \frac{e^{-s}}{s(s^2+1)} - \frac{s}{s^2+1} + \frac{s e^{-s}}{s^2+1} + \frac{1}{s} - \frac{e^{-s}}{s} \end{cases}$$

$$\Rightarrow \begin{cases} y_1(t) = -1 + \cos t + \sin t + u(t-1)\cdot\left[1 - \cos(t-1) - \sin(t-1)\right] \\ y_2(t) = 1 - \cos t + \sin t + u(t-1)\cdot\left[-1 + \cos(t-1) - \sin(t-1)\right]. \end{cases}$$

$$\left(\mathcal{L}\left(\frac{1}{\omega^2}(1-\cos\omega t)\right) = \frac{1}{s(s^2+\omega^2)} \;;\; \mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}F(s) \;;\; \mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2} \right.$$
$$\left. \mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2} \right)$$

4) a)

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{bmatrix} \cdot 2 \sim \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{bmatrix} \left(-\frac{3}{4}\right) \sim \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & 10 & \frac{125}{2} \end{bmatrix}$$

takaisinsijoituksella $x_1 = \frac{19}{16}$, $x_2 = \frac{63}{16}$, $x_3 = -\frac{25}{4}$.

b)

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 0 & 5 \\ 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \end{bmatrix} \left(-\frac{2}{3}\right) \sim \begin{bmatrix} 3 & 2 & 0 & 5 \\ 0 & -\frac{4}{3} & -1 & -\frac{4}{3} \\ 0 & 4 & 3 & 8 \end{bmatrix} \cdot 3$$

$$\sim \begin{bmatrix} 3 & 2 & 0 & 5 \\ 0 & -4/3 & -1 & -4/3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \Rightarrow 0 = 4$$

inkonsistentti, ei ratkaisua.

5)

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \cdot 2 \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$$

Jotta ratkaisuja, tulee olla $k + 2g + h = 0$.

$h = k = 1 \Rightarrow g = -1$. Takaisinsijoittamalla

$x_2 = 5x_3 + 1$

$x_1 = 4x_2 - 7x_3 - 1 = 13x_3 + 3$

$x_3 \in \mathbb{R}$.