LASKUHARJOITUS 4 LV K3/P3 SYKSY 2005

① $(\mathcal{L}f')(s) = sF(s) - f(0)$, $(\mathcal{L}f'')(s) = s^2F(s) - sf(0) - f'(0)$

$D\sinh t = \cosh t$, $D\cosh t = \sinh t$

$\mathcal{L}(\sinh t) = \mathcal{L}((\sinh t)'') = s^2\mathcal{L}(\sinh t) - s\sinh(0) - \cosh(0)$

$\mathcal{L}(\sinh t) = s^2\mathcal{L}(\sinh t) - 1$, koska $\sinh(0) = 0$
$\cosh(0) = 1$

$\mathcal{L}(\sinh t)(1-s^2) = -1$

$$\mathcal{L}(\sinh t) = \frac{-1}{(1-s^2)} = \frac{1}{(s^2-1)}$$

$\mathcal{L}$-muunnos suppenee, kun $\operatorname{Re} s > 1$

$\mathcal{L}(\cosh t) = \mathcal{L}((\cosh t)'') = s^2\mathcal{L}(\cosh t) - s\cosh(0) - \sinh(0)$

$\mathcal{L}(\cosh t) = s^2\mathcal{L}(\cosh t) - s$

$\mathcal{L}(\cosh t)(1-s^2) = -s$

$$\mathcal{L}(\cosh t) = \frac{-s}{(1-s^2)} = \frac{s}{(s^2-1)}$$

$\mathcal{L}$-muunnos suppenee, kun $\operatorname{Re} s > 1$

(2) $y'' + 2y' + 5y = 2\cos t \quad , \quad y(0) = 0 \; , \; y'(0) = 0$

1. Homogeeninen yhtälö $y'' + 2y' + 5y = 0$

yrite $y_h(t) = e^{\lambda t}$

sijoitus: $\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 5 e^{\lambda t} = 0$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\frac{-2 \pm \sqrt{4 - 4\cdot 1 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm \sqrt{-4} = -1 \pm 2i$$

HY:n ratkaisu: $y_h(t) = e^{-t}\left(C_1 \cos(2t) + C_2 \sin(2t)\right)$

2. Etsitään jokin EHY:n ratkaisu yritteellä $y_e = a\cos t + b\sin t$

~~sijoitus:~~ $\cancel{-a\cos t} - b\sin t\; \cancel{+2(}-a\sin t + b\cancel{\cos t}) + 5(a\cos t + b\sin t) = 2\cos t$

$\cancel{(-a + 2b + 5a)\cos t + (-b - 2a + 5b)\sin t = 2\cos t}$

$$4a + 2b = 2 \quad \& \quad 4b - 2a = 0$$

$$\Rightarrow \; b = \frac{1}{5} \; , \; a = \frac{2}{5} \quad \Rightarrow y_e = \frac{2}{5}\cos t + \frac{1}{5}\sin t$$

nyt EHY:n yleinen on $y = y_h + y_e$

Vakiot C_1 ja C_2 saadaan ratkaistua alkuehtojen perusteella

$$y(t) = e^{-t}\left(C_1\cos(2t) + C_2\sin(2t)\right) + \frac{2}{5}\cos t + \frac{1}{5}\sin t$$

$$y(0) = 1\cdot(C_1 + 0) + \frac{2}{5} + 0 = 0 \quad \Rightarrow C_1 = -\frac{2}{5}$$

$$y'(t) = -e^{-t}\left(C_1\cos(2t) + C_2\sin(2t)\right) + e^{-t}\left(-2C_1\sin(2t) + 2C_2\cos(2t)\right) - \frac{2}{5}\sin t + \frac{1}{5}\cos t$$

$$y'(0) = -1(C_1 + 0) + 1(0 + 2C_2) + \frac{1}{5} = 0 \quad \Rightarrow 2C_2 = C_1 - \frac{1}{5} \quad \Rightarrow C_2 = -\frac{3}{10}$$

$$y(t) = \frac{2}{5}\cos t + \frac{1}{5}\sin t - \frac{2}{5}e^{-t}\cos(2t) - \frac{3}{10}e^{-t}\sin(2t)$$

(3) $y'' + 2y' + 5y = 2\cos t$, $y(0) = 0$, $y'(0) = 0$

Siirrytään t-avaruudesta s-avaruuteen tekemällä kullekin termille $\mathcal{L}$-muunnos:

$$s^2 Y - s\underbrace{f(0)}_{=0} - \underbrace{f'(0)}_{=0} + 2sY - 2\underbrace{f(0)}_{=0} + 5Y = 2 \cdot \frac{s}{s^2+1}$$

$$Y(s) \cdot (s^2 + 2s + 5) = \frac{2s}{s^2+1}$$

$$Y = \frac{2s}{(s^2+2s+5)(s^2+1)}$$ tehdään osamurtohajotelma

$$Y = \frac{A + Bs}{(s^2+2s+5)} + \frac{C + Ds}{(s^2+1)}$$

$$= \frac{As^2 + A + Bs^3 + Bs + Cs^2 + 2Cs + 5C + Ds^3 + 2Ds^2 + 5Ds}{(s^2+2s+5)(s^2+1)}$$

$[s^3]$: $B + D = 0$ $\Rightarrow B = -D$
$[s^2]$: $A + C + 2D = 0$
$[s]$: $B + 2C + 5D = 2$
$[s^0]$: $A + 5C = 0$ $\Rightarrow A = -5C$

$A + C + 2D = -5C + C + 2D = 0 \Rightarrow D = 2C$

$B + 2C + 5D = B - B - 5B = 2 \Rightarrow B = -\frac{2}{5},\ D = \frac{2}{5},\ C = \frac{1}{5},\ A = -1$

$$Y = \frac{-1 - \frac{2}{5}s}{(s^2+2s+5)} + \frac{\frac{1}{5} + \frac{2}{5}s}{(s^2+1)}$$

täydennetään ensimmäisen termin nimittäjä neliöksi.
Huomaa muutos osoittajassa shiftauksen vuoksi.

$$Y = \frac{-1 - \frac{2}{5}(s+1) + \frac{2}{5}}{(s+1)^2 + 4} + \frac{\frac{1}{5}}{(s^2+1)} + \frac{\frac{2}{5}s}{(s^2+1)}$$

$$Y = -\frac{3}{5}\,\frac{1}{(s^2+1)+4} - \frac{2}{5}\,\frac{s+1}{(s+1)^2+4} + \frac{1}{5}\,\frac{1}{s^2+1} + \frac{2}{5}\,\frac{s}{s^2+1}$$

suoritetaan $\mathcal{L}^{-1}$-muunnos termeittäin, kts. taulukko

$$y = -\frac{3}{5}\cdot\frac{1}{2}\cdot e^{-t}\sin(2t) - \frac{2}{5}\cdot e^{-t}\cos(2t) + \frac{1}{5}\sin(t) + \frac{2}{5}\cos(t)$$

$$= \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t) - \frac{2}{5}e^{-t}\cos(2t) - \frac{3}{10}e^{-t}\sin(2t)$$

HUOM! ensimmäisen termin kerroin $1/2$ seuraa muunnoksesta

$$\frac{1}{\omega}\sin(\omega t) \overset{\mathcal{L}}{\Longrightarrow} \frac{1}{s^2+\omega^2}$$

4. a) $f(t) = u(t+2) - u(t+1) + u(t-1) - u(t-2)$

c) (ks. myös seuraava sivu)

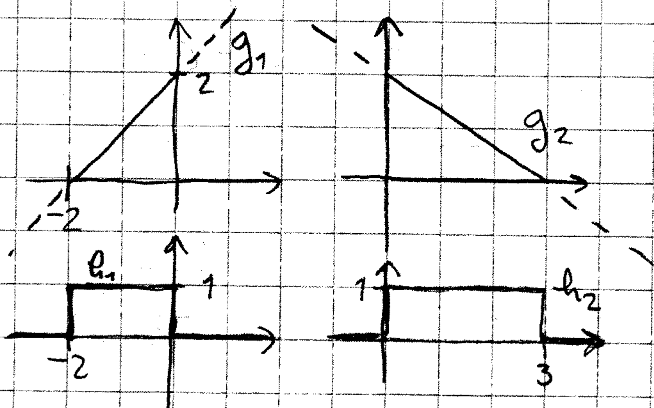
$$-u(-t+2) + \frac{1}{2}t\,u(t+2) - \frac{1}{2}t\,u(t-2) + u(t-2)$$

$$f(t) = -u(-t+2) + \frac{1}{2}t\,u(t+2) + \left(1-\frac{1}{2}t\right)u(t-2)$$

b)

$g_1(t) = t+2 \quad h_1(t) = u(t+2) - u(t)$

$g_2(t) = -\frac{2}{3}t + 2 \quad h_2(t) = u(t) - u(t-3)$



$$f(t) = \underbrace{g_1(t)}_{\text{"MITÄ TAPAHTUU"}}\underbrace{h_1(t)}_{\text{"MISSÄ TAPAHTUU"}} + g_2(t)\,h_2(t)$$

$$= (t+2)\left[u(t+2) - u(t)\right] + \left(-\frac{2}{3}t+2\right)\left[u(t) - u(t-3)\right]$$

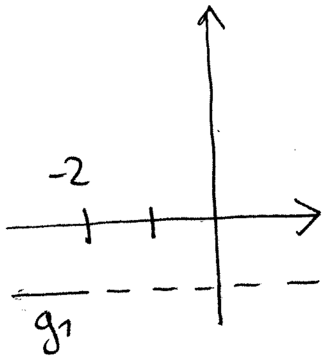
$$= (t+2)\,u(t+2) - \frac{5}{3}t\,u(t) + \left(-\frac{2}{3}t+2\right)u(t-3)$$

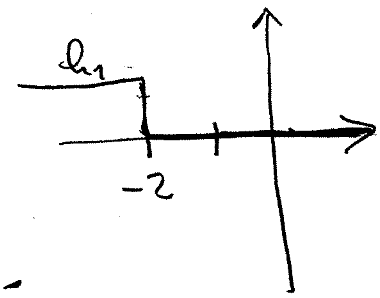
"SUORAVIIVAINEN" VAIHTOEHTO C-KOHTAAN:

4. c)

$g_1(t) = -1$

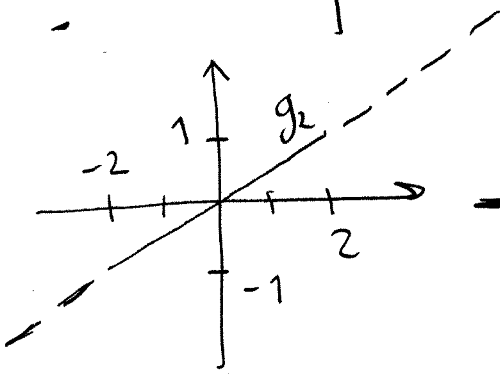
$h_1(t) = 1 - u(t+2)$

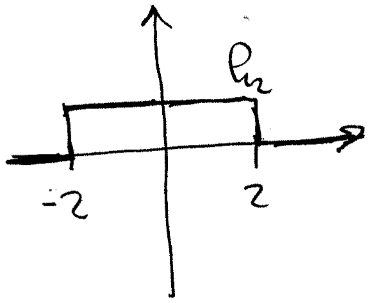




$g_2(t) = \frac{t}{2}$

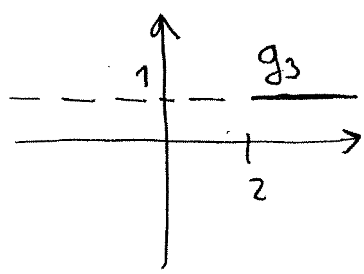
$h_2(t) = u(t+2) - u(t-2)$

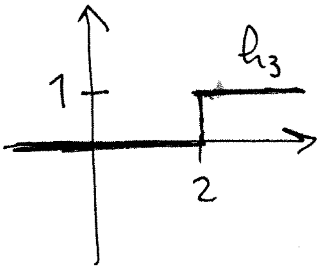




$g_3(t) = 1$

$h_3(t) = u(t-2)$





$$f(t) = h_1(t)g_1(t) + h_2(t)g_2(t) + h_3(t)g_3(t)$$

$$= -1 + u(t+2) + \frac{t}{2}u(t+2) - \frac{t}{2}u(t-2) + u(t-2)$$

$$= -1 + \left(1+\frac{t}{2}\right)u(t+2) + \left(1-\frac{t}{2}\right)u(t-2)$$

⑤ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$

a) $F(s) = \frac{e^{-5s}}{s} = e^{-5s}\cdot\frac{1}{s}$ $\quad f(t) = 1\cdot u(t-5) = \underline{u(t-5)}$

b) $F(s) = \frac{e^{-3s}}{(s-2)^2} = e^{-3s}\cdot\frac{1}{(s-2)^2}$. $\quad \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right) = te^{2t}$

niin $\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{(s-2)^2}\right\} = (t-3)e^{2(t-3)}u(t-3) = \underline{\left(e^{2t-6}t - 3e^{2t-6}\right)u(t-3)}$

c) $\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{s}{(s^2+9)}\right\}$

$G(s) = \frac{s}{s^2+9} \overset{\mathcal{L}^{-1}}{\Rightarrow} g(t) = \cos 3t$

$\mathcal{L}^{-1}\left\{e^{-2s}\frac{s}{s^2+9}\right\} = \cos(3(t-2))\,u(t-2) = \underline{\cos(3t-6)u(t-2)}$

⑥ $y'' + 2y' = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$, $y(0) = 0$, $y'(0) = 1$

askelfunktion avulla $\quad y'' + 2y' = u(t-1)$, $y(0) = 0$, $y'(0) = 1$

Siirrytään s-tasoon $\mathcal{L}$-muunnoksen avulla kuten tehtävässä 3.

$s^2Y - sy(0) - y'(0) + 2sY - 2y(0) = \frac{e^{-s}}{s}$

$Y\cdot(s^2+2s) - 1 = \frac{e^{-s}}{s}$

$Y\cdot(s^2+2s) = \frac{e^{-s}}{s} + 1$

$Y = \frac{e^{-s}}{s(s^2+2s)} + \frac{1}{(s^2+2s)} = \frac{e^{-s}}{s^2(s+2)} + \frac{1}{s(s+2)}$

Molemmille termeille osamurtokehitelmät

$$\frac{1}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2} = \frac{As + 2A + Bs^2 + 2Bs + Cs^2}{s^2(s+2)}$$

$$\Rightarrow \begin{aligned} B + C &= 0 \\ A + 2B &= 0 \\ 2A &= 1 \end{aligned} \Rightarrow A = \frac{1}{2},\ B = -\frac{1}{4},\ C = \frac{1}{4}$$

$$\frac{1}{s(s+2)} = \frac{D}{s} + \frac{E}{s+2} = \frac{Ds + 2D + Es}{s(s+2)}$$

$$D = \frac{1}{2},\ E = -\frac{1}{2}$$

$$\mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left[\frac{1}{4}\left(\frac{2e^{-s}}{s^2} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s+2}\right) + \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)\right]$$

tarvitaan muunnoksia $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$, $\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$, $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$

kertominen e^{-s}:llä vastaa t-siirtoa 1:llä

$$y(t) = \frac{1}{4}\left[2(t-1)u(t-1) - u(t-1) + e^{-2(t-1)}u(t-1)\right] + \frac{1}{2}\left[1 - e^{-2t}\right]$$

$$y(t) = -\frac{1}{2}e^{-2t} + \frac{1}{2} + u(t-1)\left(\frac{1}{2}t - \frac{1}{2} - \frac{1}{4} + \frac{1}{4}e^{-2t+2}\right)$$

$$= -\frac{1}{2}e^{-2t} + \frac{1}{2} + u(t-1)\left(\frac{1}{2}t - \frac{3}{4} + \frac{1}{4}e^{-2t+2}\right)$$