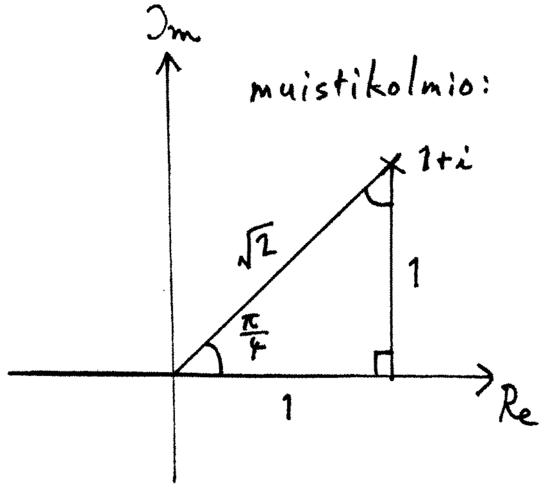
Harj 3 LV (viikko 39) loppuviikko malliratkaisut

1. a) $(1+i)^{i-1} = ?$

merkitään: $z = 1+i$

$\rightarrow$ $|z| = \sqrt{2}$ $\quad$ $\operatorname{Arg} z = \frac{\pi}{4}$

muistikolmio:



polaarimuodossa:

$$z = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

nyt:

$$z^{i-1} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{i-1}$$

$$= \left(\sqrt{2}\right)^{i-1} \cdot \left(e^{i\frac{\pi}{4}}\right)^{i-1} = \left(e^{\ln\sqrt{2}}\right)^{i-1} \cdot e^{-\frac{\pi}{4} - \frac{\pi}{4} i} \qquad i^2 = -1$$

$$= e^{i\ln\sqrt{2} - \ln\sqrt{2}} \cdot e^{-\frac{\pi}{4}} \cdot e^{-\frac{\pi}{4} i}$$

$$= \underbrace{e^{-\frac{\pi}{4} - \ln\sqrt{2}}}_{r} \cdot e^{\overbrace{\left(\ln\sqrt{2} - \frac{\pi}{4}\right) i}^{\varphi}}$$

$$\approx 0{,}32\, e^{-0{,}44 i}$$

$$\approx 0{,}2919 - 0{,}1370 i$$

b)

$$\sin z = \cosh 2$$

tiedetään (Kreyszig):

$\cosh z = \cos iz$ $\quad$ (1)

$\cos z = \sin\left(\frac{\pi}{2} - z\right)$ $\quad$ (2)

$\sin z = \sin w \leftrightarrow$ $\quad$ (3)

$z = w + n2\pi \vee z = \pi - w + n2\pi,$
$n \in \mathbb{Z}$

$\leftrightarrow \sin z = \cos(2i)$ $\quad$ (1)

$\leftrightarrow \sin z = \sin\left(\frac{\pi}{2} - 2i\right)$ $\quad$ (2)

$$\leftrightarrow z = \frac{\pi}{2} - 2i + n2\pi \vee z = \pi - \left(\frac{\pi}{2} - 2i\right) + n2\pi, \quad n \in \mathbb{Z} \quad (3)$$

siis: $\quad z = \frac{\pi}{2} + n2\pi \pm 2i, \quad n \in \mathbb{Z}$

$$= \tfrac{1}{2}(4n+1)\pi \pm 2i, \quad n \in \mathbb{Z}$$

2. CR-yhtälöt: $u_x = v_y$, $u_y = -v_x$

a) $v(x,y) = 2y(x+1) = 2xy + 2y$

$$v_x = 2y \xrightarrow[CR]{} u_y = -2y$$

$$v_y = 2x+2 \longrightarrow u_x = 2x+2$$

nyt:

$$u_x = 2x+2$$

$$\Leftrightarrow \frac{\partial u}{\partial x} = 2x+2 \xrightarrow{\int} u = x^2 + 2x + C(y)$$

ja $u_y = -2y$

$$\Leftrightarrow \frac{\partial u}{\partial y} = -2y \xrightarrow{\int} u = -y^2 + D(x)$$

$$\longrightarrow C(y) = -y^2 + C_1 \quad D(x) = x^2 + 2x + D_1$$

siis

$$u = x^2 + 2x - y^2 + C$$

lisäksi osittaisderivaatat ovat jatkuvia siis

$$T: f(z) = x^2 + 2x - y^2 + i(2y(x+1)) + C$$

$$= x^2 - y^2 + 2ixy + i(2x + 2y) + C$$

$$= \underline{\underline{z^2 + 2z + C}} \qquad (C \in \mathbb{R})$$

2.

b)

$$v(x,y) = e^{(x^2-y^2)} \cdot \sin(2xy)$$

$$v_x = 2x e^{(x^2-y^2)} \sin(2xy) + 2y \cos(2xy) e^{(x^2-y^2)}$$

$$\overset{CR}{=} -u_y \quad (1)$$

$$v_y = -2y e^{(x^2-y^2)} \sin(2xy) + 2x \cos(2xy) e^{(x^2-y^2)}$$

$$\overset{CR}{=} u_x \quad (2)$$

valistunut arvaus: $u(x,y) = e^{(x^2-y^2)} \cdot \cos(2xy)$

tarkistus:

$$u_x = 2x e^{(x^2-y^2)} \cos(2xy) - 2y \sin(2xy) e^{(x^2-y^2)} \quad \underset{\text{VRT (1)}}{\text{ok!}}$$

$$u_y = -2y e^{(x^2-y^2)} \cos(2xy) - 2x \sin(2xy) e^{(x^2-y^2)} \quad \underset{\text{VRT (2)}}{\text{ok!}}$$

lisäksi osittaisderivaatat ovat jatkuvia siis

$$T: f(z) = e^{(x^2-y^2)} \cos(2xy) + i\left(e^{(x^2-y^2)} \sin(2xy)\right)$$

$$= e^{(x^2-y^2)} \left[\cos 2xy + i \sin 2xy\right]$$

$$= e^{x^2-y^2+2ixy} = \underline{\underline{e^{z^2}}}$$

Tässäkin tulokseen voi toki lisätä reaalisen vakion:

$$\underline{\underline{f(z) = e^{z^2} + C,}} \quad C \in \mathbb{R}$$

3.

$$\boxed{z = re^{i\varphi}}$$

$$f(z) = \frac{1}{z^2} = \frac{1}{(re^{i\varphi})^2} = \frac{1}{r^2 e^{i2\varphi}} = r^{-2} e^{-i2\varphi}$$

$$\boxed{\text{Eulerin kaava: } e^{i\varphi} = \cos\varphi + i\sin\varphi}$$

$$= r^{-2}\left(\cos(-2\varphi) + i\sin(-2\varphi)\right) \qquad \boxed{\begin{array}{l}\cos(-z) = \cos z \\ \sin(-z) = -\sin z\end{array}}$$

$$= \underbrace{r^{-2}\cos(2\varphi)}_{u(r,\varphi)} + i\underbrace{\left(-r^{-2}\sin(2\varphi)\right)}_{v(r,\varphi)}$$

analyyttisyys: CR-yhtälöt napakoordinaateissa:

$$u_r = \frac{1}{r} v_\varphi, \quad v_r = -\frac{1}{r} u_\varphi$$

$$u(r,\varphi) = r^{-2}\cos(2\varphi) \qquad v(r,\varphi) = -r^{-2}\sin(2\varphi)$$

$$u_r = -2r^{-3}\cos(2\varphi)$$

NYT:

$$\frac{1}{r} v_\varphi = \frac{1}{r}\left(-r^{-2} 2\cos(2\varphi)\right) = u_r \quad \text{CR ok!}$$

JA:

$$v_r = 2r^{-3}\sin(2\varphi)$$

SEKÄ:

$$-\frac{1}{r} u_\varphi = -\frac{1}{r}\left(r^{-2}\cdot\left(-2\sin(2\varphi)\right)\right) = v_r \text{ CR OK!}$$

osittaisderivaatat jatkuvia PAITSI, kun $r = 0$ (origo)

SIIS f on analyyttinen

3. jatkuu PRUJUISTA TIEDETÄÄN!

$$f'(z) = e^{-i\varphi}(u_r + i v_r) = \frac{e^{-i\varphi}}{r}(v_\varphi - i u_\varphi)$$

NYT esim.

$$u_r = -2r^{-3}\cos(2\varphi), \quad v_r = 2r^{-3}\sin(2\varphi)$$

$$f'(z) = e^{-i\varphi}\left(\left(-2r^{-3}\cos(2\varphi)\right) + i\left(2r^{-3}\sin(2\varphi)\right)\right)$$

$$= -2r^{-3}\left[e^{-i\varphi}\left(\cos(2\varphi) - i\sin(2\varphi)\right)\right]$$

sin & cos kompleksiset määritelmät

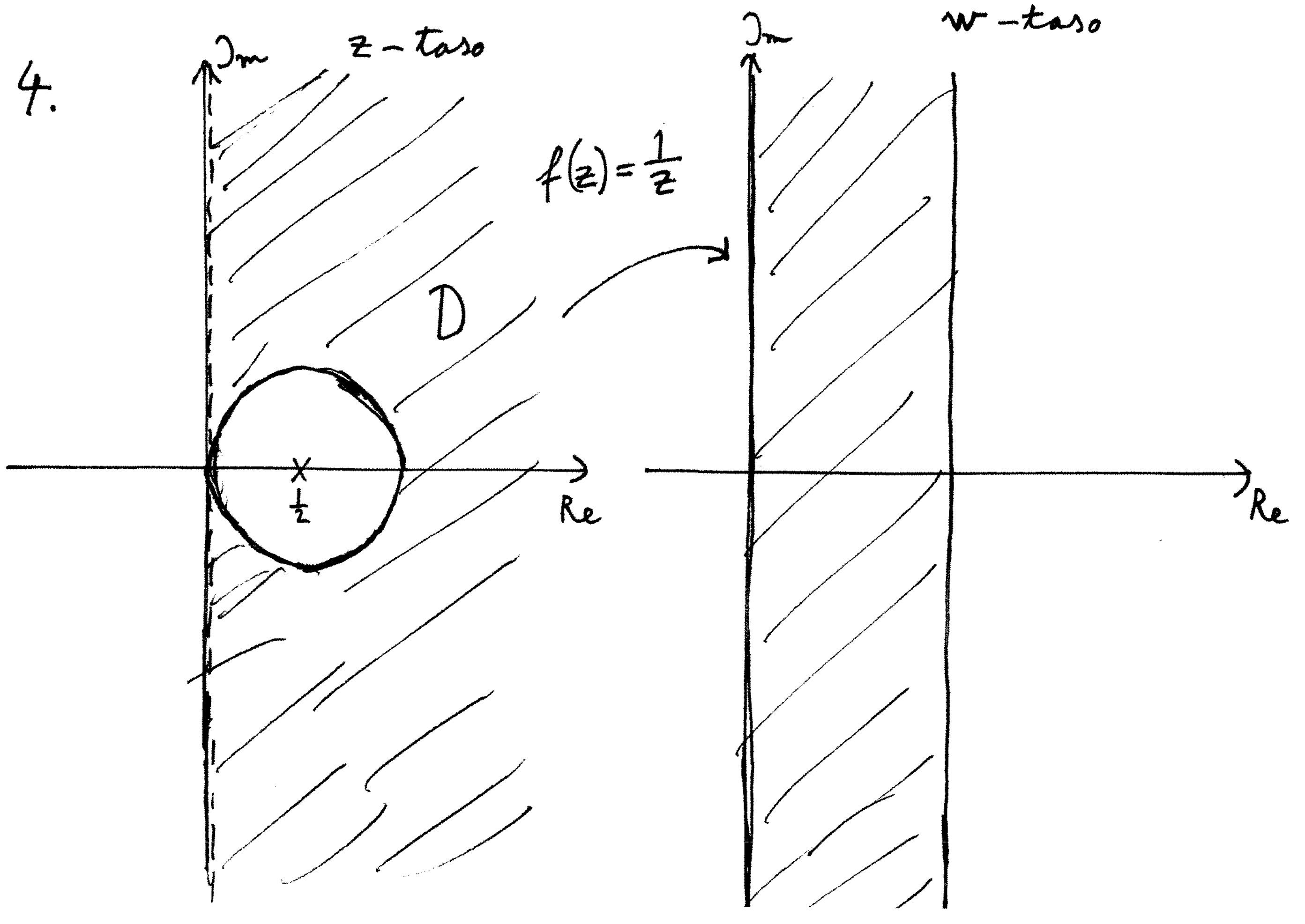
$$= -2r^{-3}\left[e^{-i\varphi}\left(\tfrac{1}{2}(e^{i2\varphi} + e^{-i2\varphi}) - i\cdot\tfrac{1}{2i}(e^{i2\varphi} - e^{-i2\varphi})\right)\right]$$

$$= -2r^{-3}\left[e^{-i\varphi}\left(\tfrac{1}{2}e^{i2\varphi} + \tfrac{1}{2}e^{-i2\varphi} - \tfrac{1}{2}e^{i2\varphi} + \tfrac{1}{2}e^{-i2\varphi}\right)\right]$$

$$= -2r^{-3}\left(e^{-i\varphi}\cdot e^{-i2\varphi}\right) = -2r^{-3}e^{-i3\varphi}$$

$$= -2\frac{1}{r^3 e^{i3\varphi}} = -2\frac{1}{z^3} \quad \text{OK!}$$

4.



Tutkitaan vihjeen mukaan:

$$f(z) = \frac{1}{z} = \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}\,, \qquad \underline{z \neq 0}$$

$$\mathrm{Re}\left\{\frac{1}{z}\right\} = \frac{x}{x^2+y^2} = c \iff \frac{x}{c} = x^2+y^2$$

$$0 < c \leq 1$$

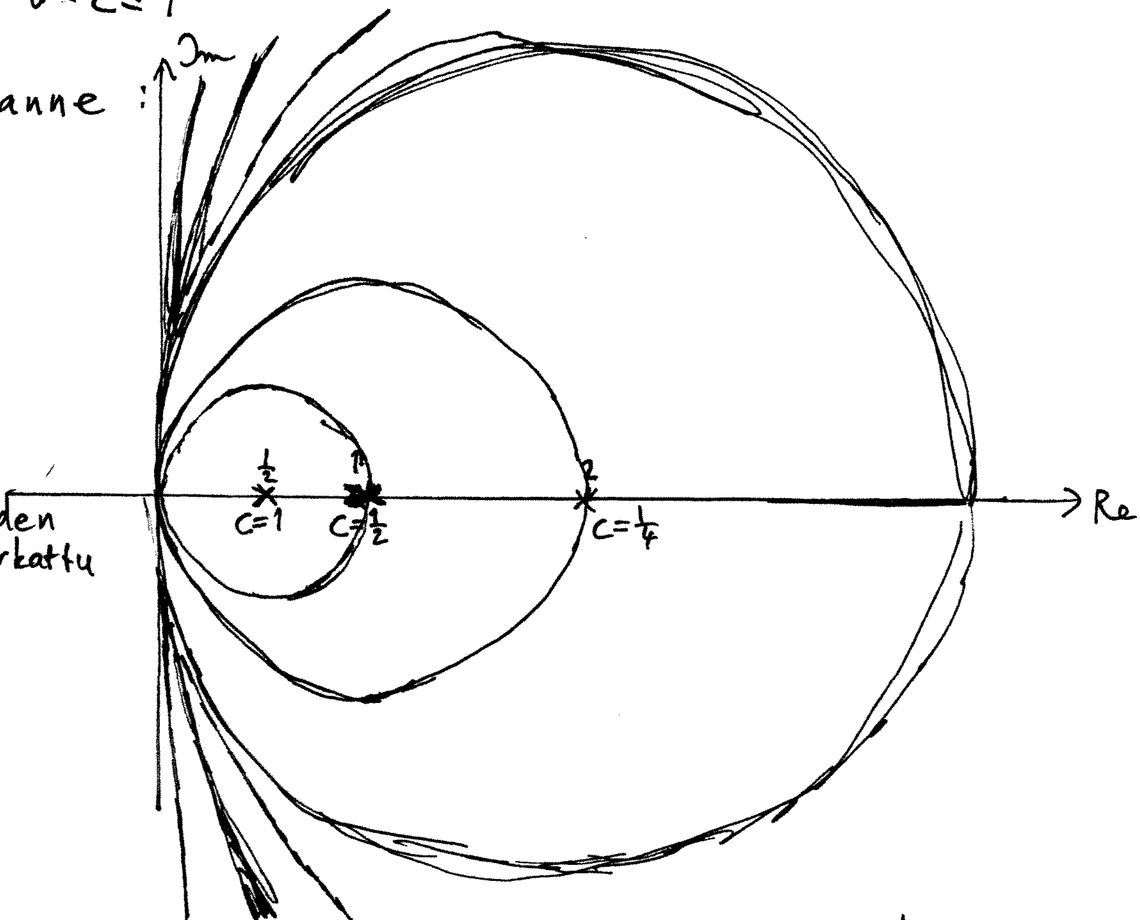
$$\iff x^2 - \frac{x}{c} + y^2 = 0 \iff x^2 - 2\cdot x\cdot\frac{1}{2c} + \left(\frac{1}{2c}\right)^2 + (y-0)^2 = \left(\frac{1}{2c}\right)^2$$

$$\iff \left(x-\frac{1}{2c}\right)^2 + (y-0)^2 = \left(\frac{1}{2c}\right)^2$$ ympyrä kp: $\left(\frac{1}{2c}, 0\right)$ r: $\frac{1}{2c}$

JOSTA POISTETTU PISTE $(0,0)$ (KOSKA $z \neq 0$)

nämä ympyrät ilmaisevat koko alueen D, kun $0 < c \leq 1$!

tilanne:



keskipisteiden c-arvot merkattu

Siis alue D kuvautuu kuvauksessa $f(z) = \frac{1}{z}$ alueeseen $0 < \mathrm{Re}\, w \leq 1$. ∎