K3/P3 2005 laskuharjoituksen 1 malliratkaisut

Loppuviikko 37

1. (a) $(z_1 - \overline{z_2})^2 = (3+4i-(5+2i))^2$

$$= (-2+2i)\cdot(-2+2i)$$

$$= 4 - 4i - 4i + 4i^2 = \underline{\underline{-8i}}$$

$$\frac{z_1}{z_2} = \frac{3+4i}{5-2i} = \frac{(3+4i)(5+2i)}{(5-2i)(5+2i)}$$

$$= \frac{15+6i+20i-8}{5^2+2^2} = \frac{7+26i}{29}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \underline{\underline{\frac{7}{29} - \frac{26}{29}i}}$$

(b) $\frac{z}{\overline{z}} = \frac{z^2}{z\overline{z}} = \frac{(x+iy)^2}{(x+iy)(x-iy)} = \frac{x^2-y^2+2ixy}{x^2+y^2}$

$$\operatorname{Re}\frac{z}{\overline{z}} = \underline{\underline{\frac{x^2-y^2}{x^2+y^2}}}$$

$$\operatorname{Im}\frac{z}{\overline{z}} = \underline{\underline{\frac{2xy}{x^2+y^2}}}$$

HUOM! Tässä näkyy myös laskusäännön $|z| = \sqrt{z\overline{z}}$ johto.

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2.

$$(\cos\varphi + i\sin\varphi)^3 = \cos 3\varphi + i\sin 3\varphi$$

$$\Rightarrow \cos 3\varphi = Re\left[(\cos\varphi + i\sin\varphi)^3\right]$$

$$= Re\left[\cos^3\varphi + 3\cos^2\varphi\, i\sin\varphi + 3\cos\varphi\, i^2\sin^2\varphi + i^3\sin^3\varphi\right]$$

$$= Re\left[\cos^3\varphi + 3i\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi - i\sin^3\varphi\right]$$

$$= \underline{\underline{\cos^3\varphi - 3\cos\varphi\sin^2\varphi}}$$

Vastaavasti

$$\sin 3\varphi = Im\left[(\cos\varphi + i\sin\varphi)^3\right]$$

$$= \underline{\underline{3\cos\varphi^2\sin\varphi - \sin^3\varphi}}$$

(Tässä laskussa oletettu, että $\varphi \in \mathbb{R} \Rightarrow \cos\varphi, \sin\varphi, \cos 3\varphi, \sin 3\varphi \in \mathbb{R}$.)

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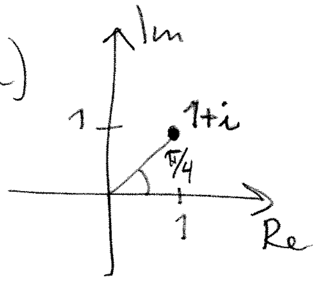
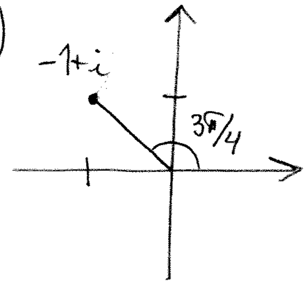
3.

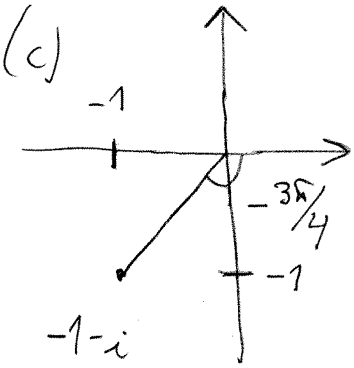
(a)



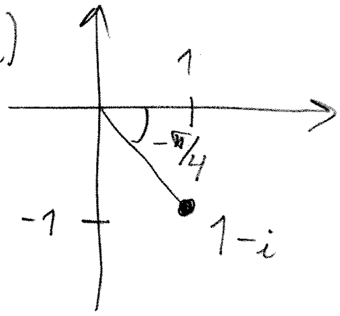
(b)



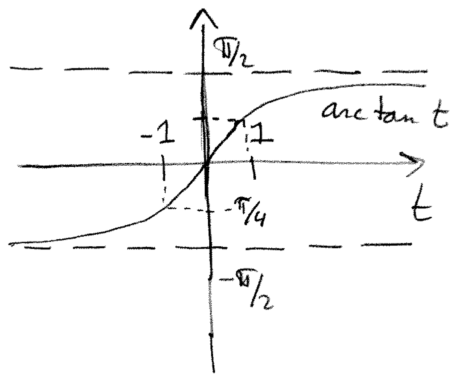
(c)



(d)



(a) $\frac{\pi}{4} = \underbrace{\overline{\arctan}\,\frac{1}{1}}_{=\pi/4} + 0\pi$



(b) $\frac{3\pi}{4} = \underbrace{\overline{\arctan}\,\frac{1}{-1}}_{=-\pi/4} + 1\pi$

(c) $-\frac{3\pi}{4} = \underbrace{\overline{\arctan}\,\frac{-1}{-1}}_{=\pi/4} - 1\pi$

(d) $-\frac{\pi}{4} = \underbrace{\overline{\arctan}\,\frac{-1}{1}}_{=-\pi/4} + 0\pi$

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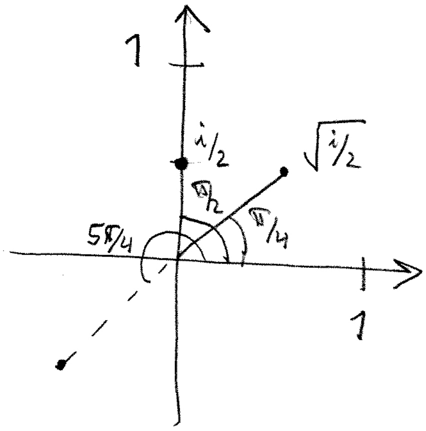
4.

$$\left|\frac{i}{2}\right| = \frac{1}{2} \qquad \operatorname{Arg}\frac{i}{2} = \frac{\pi}{2}$$



$$\sqrt{\frac{i}{2}} = \sqrt[2]{\frac{i}{2}} = w \quad \text{missä}$$

$$|w| = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

VALITTU SITEN, ETTÄ TULOS $\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\operatorname{Arg} w = \frac{\operatorname{Arg}\frac{i}{2}}{2} + 0 \cdot \frac{2\pi}{2} = \frac{\pi}{4}$$

Siis $\sqrt{\frac{i}{2}} = w = \frac{1}{\sqrt{2}}\left(\underbrace{\cos\frac{\pi}{4}}_{=\frac{1}{\sqrt{2}}} + i\underbrace{\sin\frac{\pi}{4}}_{=\frac{1}{\sqrt{2}}}\right) = \frac{1+i}{2}$.

$\square$

Lisätietoa: Kuten sanottu, neliöjuurella (= toisella juurella) on 2 erillistä arvoa. Toiselle arvolle w_1

$$\arg w_1 = \frac{\operatorname{Arg}\frac{i}{2}}{2} + 1 \cdot \frac{2\pi}{2} = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

$$\Rightarrow w_1 = \frac{1}{\sqrt{2}}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{-1-i}{2} = -w.$$

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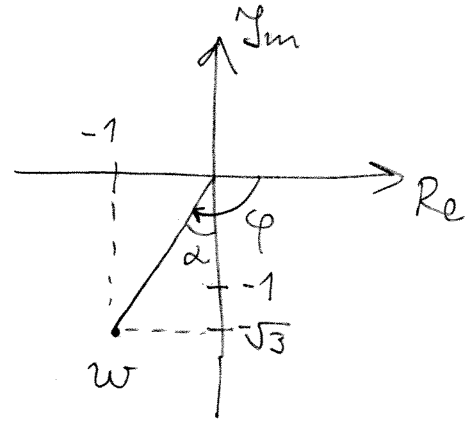
(5.)

$$z^5 = -1 - i\sqrt{3} =: w$$

$$|w| = \sqrt{1^2 + \sqrt{3}^2} = 2$$



$$\operatorname{Arg} w = \varphi = -\alpha - \frac{\pi}{2}$$

$$= -\frac{\pi}{6} - \frac{3\pi}{6}$$

$$= -\frac{2\pi}{3}$$

Olkoon $z = r(\cos\varphi + i\sin\varphi)$.

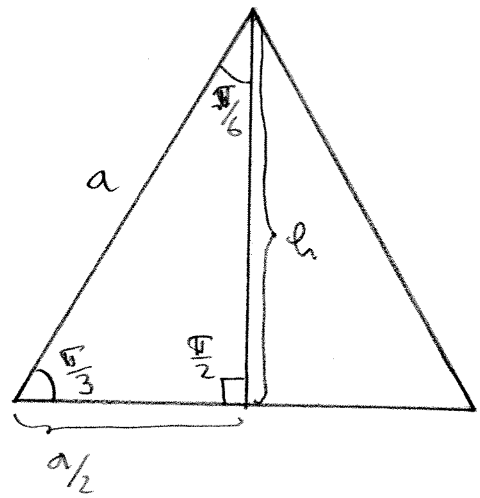
$$z^5 = -1 - i\sqrt{3}$$

$$\Leftrightarrow \begin{cases} |z|^5 = 2 \\ 5 \arg z = -\frac{2\pi}{3} + k \cdot 2\pi \end{cases}$$

$$\Rightarrow \begin{cases} |z| = \sqrt[5]{2} \\ \arg z = -\frac{2\pi}{15} + k\frac{2\pi}{5}, \quad k \in \{0,1,2,3,4\} \end{cases}$$

$$\Rightarrow \underline{\underline{z = \sqrt[5]{2}\left[\cos\left(-\frac{2\pi}{15} + k\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{15} + k\frac{2\pi}{5}\right)\right]}}$$

Likiarvot: $1{,}049 - 0{,}467i$, $0{,}769 + 0{,}854i$, $-0{,}574 + 0{,}995i$, $-1{,}124 - 0{,}239i$, $-0{,}120 - 1{,}142i$



TASASIVUINEN KOLMIO

$$\left(\frac{a}{2}\right)^2 + h^2 = a^2$$

JOS NYT $h = \sqrt{3}$

$$\frac{a^2}{4} + 3 = a^2$$

$$\Rightarrow \frac{3}{4}a^2 = 3$$

$$\Rightarrow a^2 = 4 \Rightarrow a = 2$$

$$\Rightarrow a = 1$$

ELI KOORDINAATISTOSSA OLEVA KOLMIO ON TASASIVUISEN KOLMION PUOLIKAS

$$\Rightarrow \alpha = \frac{\pi}{6}$$

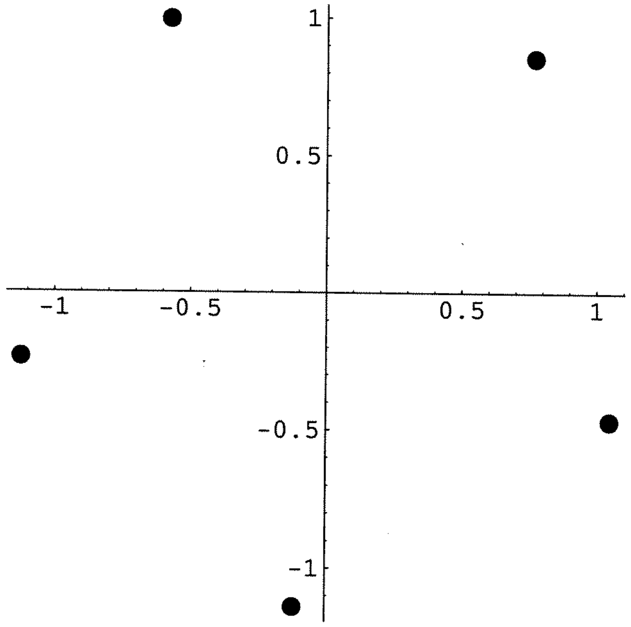
```
In[25]:= sol = Solve[z^5 == -1 - Sqrt[3] I, z]
```

Out[25]= $\{\{z \to (-1 - i\sqrt{3})^{1/5}\}, \{z \to -(-1)^{1/5}(-1 - i\sqrt{3})^{1/5}\}, \{z \to (-1)^{2/5}(-1 - i\sqrt{3})^{1/5}\}, \{z \to -(-1)^{3/5}(-1 - i\sqrt{3})^{1/5}\}, \{z \to (-1)^{4/5}(-1 - i\sqrt{3})^{1/5}\}\}$

```
In[22]:= N[sol]
```

Out[22]= $\{\{z \to 1.04939 - 0.467218\, i\}, \{z \to -1.1236 - 0.238828\, i\}, \{z \to 0.768629 + 0.853649\, i\}, \{z \to -0.120072 - 1.14241\, i\}, \{z \to -0.574349 + 0.994802\, i\}\}$

```
In[23]:= ListPlot[{Re[z], Im[z]} /. %2, PlotStyle -> PointSize[0.03], AspectRatio -> 1];
```



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6. (a) $|\omega| = \sqrt[n]{1} = 1$ ← MONITULKINTAINEN MERKINTÄ, TÄSSÄ TAVALLINEN POSITIIVISEN REAALILUVUN JUURI

$$\operatorname{Arg}\omega = \frac{\operatorname{Arg} 1}{n} + 1\cdot\frac{2\pi}{n} = \frac{2\pi}{n}$$

Siis $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Minkälaisia ovat 1:n juuret?

Olkoon $z = r(\cos\varphi + i\sin\varphi)$ 1:n n:s juuri, ts.

$$z^n = r^n(\cos n\varphi + i\sin n\varphi) = 1 = 1(\cos 0 + i\sin 0)$$

$$\Leftrightarrow \begin{cases} r^n = 1 \Leftrightarrow r = 1 \\ n\varphi = 0 + k\cdot 2\pi \Leftrightarrow \varphi = k\frac{2\pi}{n}, \quad k\in\mathbb{N} \end{cases}$$

Kun $k=0$, saadaan $z = 1\cdot(\cos 0 + i\sin 0) = 1 \quad (=\omega^0)$

Kun $k=1$, — " — $z = 1\cdot\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right) = \omega$

Kun $k=2$, — " — $z = 1\cdot\left(\cos 2\cdot\frac{2\pi}{n} + i\sin 2\frac{2\pi}{n}\right) = \omega^2$

⋮

Kun $k=n-1$, — " — $z = 1\cdot\left(\cos(n-1)\frac{2\pi}{n} + i\sin(n-1)\frac{2\pi}{n}\right) = \omega^{n-1}$.

Mutta kun $k=n$, — " — $z = 1\cdot\Big(\underbrace{\cos \overbrace{n\cdot\frac{2\pi}{n}}^{=2\pi}}_{=1} + i\underbrace{\sin \overbrace{n\cdot\frac{2\pi}{n}}^{=2\pi}}_{=0}\Big) = 1$

ja kierros alkaa alusta.

6... (a...) [Jos halutaan olla kovin täsmällisiä, tämä voidaan perustella seuraavasti: kun $k \geq n$, voidaan kirjoittaa $k = n+m$, $m \in \mathbb{N}$. Tällöin

$$z = 1\cdot\Big(\underbrace{\cos \underbrace{(n+m)\tfrac{2\pi}{n}}_{=2\pi+m\frac{2\pi}{n}}}_{=\cos m\frac{2\pi}{n}} + i\,\underbrace{\sin \underbrace{(n+m)\tfrac{2\pi}{n}}_{=2\pi+m\frac{2\pi}{n}}}_{=\sin m\frac{2\pi}{n}}\Big)$$

Eli sama arvo saatiin jo aikaisemminkin arvolla m.]

(b) $$\sum_{k=0}^{n-1} \omega^k = \frac{\omega^{(n-1)+1} - 1}{\omega - 1} = \frac{\overbrace{\omega^n}^{=1} - 1}{\omega - 1} = 0$$

(c) Säännöllinen n-kulmio: