KP3 MALLIT
12 LV

I $\quad u(x,0) = f(x) = 1 - \frac{x}{\pi} \quad , x \in (0,\pi)$

$u_x(0,t) = 0$

$u_x(\pi,t) = 0$

$u_t = c^2 u_{xx} \qquad$ yrite: $u(x,t) = X(x)\,T(t)$

$XT' = c^2 X'' T \qquad \| : c^2 X'' T'$

$\frac{X}{c^2 X''} = \frac{T}{T'} = -k^2$

1° $\quad X = -k^2 c^2 X''$

$\Rightarrow X = A\cos kcx + B\sin kcx$

$u_x(0,t) = X'(0)\,T(t) = 0$

$X'(0) = 0 \qquad \| \; X' = -Akc\sin kcx + Bkc\cos kcx$

$Bkc = 0$

$B = 0$

$u_x(\pi,t) = X'(\pi)\,T(t) = 0$

$X'(\pi) = 0$

$-Akc\sin kc\pi = 0$

$A = 0$ tai $\sin kc\pi = 0$

ei mielekästä

$kc\pi = n\pi \qquad , n \in \mathbb{N} \cup \{0\}$

$kc = n$

$k = n/c$

$\Rightarrow X_n = A_n \cos nx$

I 2° $T = -k^2 T'$

$$\Rightarrow T = Ce^{-t/k^2} \quad \| \; k = \tfrac{n}{c}$$
$$= Ce^{-tn^2/c^2}$$

$$T_n = C_n e^{-tn^2/c^2}$$

$$u_n(x,t) = X_n T_n = D_n \cos nx \; e^{-tn^2/c^2} \qquad \| \; D_n = A_n C_n$$

$$u(x,t) = \sum_{n=0}^{\infty} D_n \cos nx \, e^{-tn^2/c^2}$$

$$u(x,0) = \sum_{n=0}^{\infty} D_n \cos nx = f(x)$$
$$= \hat{f}(x), \; x \in (0,\pi)$$

|| Muodostetaan f:n parillinen jatke $\hat{f}(x) \Rightarrow \hat{f}(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (ei siniterme jä)

$$\Rightarrow D_0 + \sum_{n=1}^{\infty} D_n \cos nx = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Nyt täytyy olla $D_0 = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(x)\,dx = 1/2 \quad \left(= 2\cdot\frac{1}{2\pi} \underbrace{\int_0^{\pi} 1 - \frac{x}{\pi}\,dx}_{\frac{1}{2}\cdot\pi\cdot 1 \text{ (kolmio)}}\right)$

ja
$$D_n = a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}(x)\cos nx\,dx$$
$$= 2\cdot\frac{1}{\pi}\int_0^{\pi} \left(1-\frac{x}{\pi}\right)\cos nx\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \cos nx\,dx - \frac{2}{\pi^2}\int_0^{\pi} x\cos nx\,dx$$
$$= \frac{2}{\pi}\Big/_0^{\pi} \frac{1}{n}\sin nx - \frac{2}{\pi^2}\left[\Big/_0^{\pi} x\frac{1}{n}\sin nx - \int_0^{\pi}\frac{1}{n}\sin nx\,dx\right]$$
$$= 0 - 0 - \frac{2}{\pi^2}\left[0 - 0 - \frac{1}{n}\Big/_0^{\pi} -\frac{1}{n}\cos nx\right]$$
$$= -\frac{2}{\pi^2 n^2}(\cos n\pi - 1)$$
$$= \frac{2}{\pi n^2}\left((-1)^{n+1} + 1\right)$$

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}\left((-1)^{n+1}+1\right)\cos nx \, e^{-tn^2/c^2}$$
$$= \frac{1}{2} + \sum_{n=1,3,5,\dots} \frac{4}{n^2\pi^2}\cos nx \, e^{-\frac{n^2}{c^2}t}$$

I $\quad \lim\limits_{t \to \infty} u(x,t) = \frac{1}{2}$

koska
$$\left| \sum_{n=1,3,5,\ldots} \frac{4}{n^2\pi^2} \cos nx \, e^{-\frac{n^2}{c^2}t} \right| \leq \sum_{n=1}^{\infty} \underbrace{\frac{4}{\pi^2}}_{\leq 1} \frac{1}{n^2} \underbrace{|\cos nx|}_{\leq 1} \underbrace{e^{-\frac{n^2}{c^2}t}}_{\leq e^{-t/c^2}}$$

$$\leq e^{-t/c^2} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{SUPPENEE}} \xrightarrow{t \to \infty} 0 .$$

Teknillinen korkeakoulu
Matematiikka

Apiola/Bingham

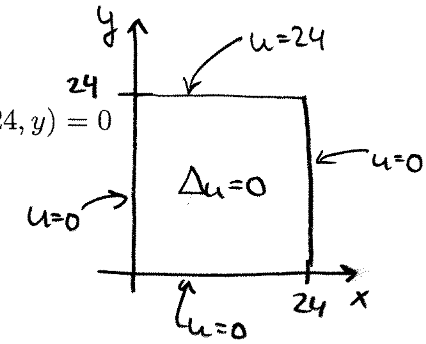
Mat-1.433/443 Matematiikan peruskurssi K3/P3 syksy 2005
`http://math.tkk.fi/teaching/k3/`

Laskuharjoitus 12 (viikko 49, 5.–9.12.2005) **Ratkaisuja**

Loppuviikko

2. Ohuen neliönmuotoisen kuparilevyn pinnat on lämpöeristetty (reunoja lukuunottamatta). Olkoon neliön sivu $a = 24$. Yläreuna pidetään 20°C-asteessa ja muut reunat 0°:ssa. Määritä tasapainolämpötilajakauma $u(x, y)$, toisin sanoen, ratkaise Laplacen yhtälö $\Delta u = 0$.

Ratkaisu:



$$u_{xx} + u_{yy} = 0, \quad u(x, 24) = 20, \quad u(0, y) = u(x, 0) = u(24, y) = 0$$

Suoritetaan muuttujien erottelu edellisen tapaan. Nyt saadaan

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -k.$$

(Voitaisiin tietysti merkitä yhtä hyvin k, jolloin osoittautuisi, että on valittava $k \leq 0$. Tulevaa ennakoiden $-k$ on vähän mukavampi.)

(x-yhtälö): $F''(x) + kF(x) = 0, \quad F(0) = F(24) = 0.$
(y-yhtälö): $G''(y) - kF(y) = 0, \quad G(0) = 0.$

x-yhtälö: Yl. ratk: $F(x) = A\cos\sqrt{k}x + B\sin\sqrt{k}x.$
$F(0) = 0 \implies A = 0, F(24) = 0 \implies \sin\sqrt{k}24 = 0 \implies k = (\frac{n\pi}{24})^2.$
Siten $F_n(x) = \sin\frac{n\pi}{24}x, \quad n \in \mathbb{N}.$

y-yhtälö: Yl. ratk.: $G(y) = Ae^{\frac{n\pi}{24}y} + Be^{-\frac{n\pi}{24}y}.$
$G(0) = 0 \implies B = -A$, joten $G(y) = 2A\sinh\frac{n\pi y}{24}$. Merk.
$G_n(y) = \sinh\frac{n\pi y}{24}.$

Siis "ominaisfunktiot" $u_n(x, y) = F_n(x)G_n(y) = \sin\frac{n\pi x}{24}\sinh\frac{n\pi y}{24}$ toteuttavat Laplacen yhtälön ja kolme 0-reunaehtoa. Samoin on laita mielivaltaisen lineaarikombinaation, jopa äärettömän, kunhan kertoimet valitaan niin, että sarja suppenee.

Yläreunaehto: $u(x, 24) = 20$

Miten on kertoimet c_n valittava, jotta $20 = \sum_{n=1}^{\infty} c_n u_n(x, y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{24}\sinh\frac{n\pi y}{24}$, kun $y = 24$. No tällöinhän on oltava $c_n \sinh n\pi = b_n$, missä b_n on $2\cdot 24$-jaksoisen parittomasti jatketun vakiofunktion sinisarjan kerroin (ts. 48-jaksoisen parittoman kanttiaallon Fourier-sarjan kerroin)

$b_n = \frac{2}{24}\int_0^{24} 20\sin\frac{n\pi x}{24}\,dx = \frac{40}{n\pi}(1 - \cos n\pi) = \frac{80}{n\pi}$, kun n on pariton, ja 0, kun n on parillinen.

Siten saadaan ratkaisu:

$$u(x, y) = \frac{80}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)\sinh((2k-1)\pi)}\sin(\frac{(2k-1)\pi x}{24})\sinh\frac{(2k-1)\pi y}{24}$$

Tai yhtä hyvin (ehkä selkeämmän näköisesti):

$$u(x, y) = \frac{80}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(\frac{n\pi x}{24})\sinh\frac{n\pi y}{24}$$

III
$$\begin{cases} u(x,0) = k\sin x - \frac{1}{2}\sin 2x \\ u_t(x,0) = 0 \\ u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

$u_{tt} = c^2 u_{xx}$ $\quad , c^2 = 1$

$u_{tt} = u_{xx}$

yrite: $u(x,t) = X(x)T(t)$

$XT'' = X''T \quad \| : XT$

$\frac{T''}{T} = \frac{X''}{X} = -p^2$

$T'' = -p^2 T$

$\Rightarrow T = A\cos pt + B\sin pt$

$u_t(x,0) = X(x)T'(0) = 0$

$T'(0) = 0 \quad \| T' = -Ap\sin pt + Bp\cos pt$

$Bp \cdot 1 = 0$

$B = 0$

$X'' = -p^2 X$

$\Rightarrow X = C\cos px + D\sin px$

$u(0,t) = X(0)\,T(t) = 0$

$X(0) = 0$

$C \cdot 1 = 0$

$C = 0$

$u(\pi,t) = X(\pi)\,T(t) = 0$

$X(\pi) = 0$

$D\sin p\pi = 0$

$D = 0$ tai $\sin p\pi = 0$

ei mielekäs

$p\pi = n\pi \quad , n \in \mathbb{N}$

$p = n$

$\Rightarrow \quad D_n \sin nx = X_n$

$A_n \cos nt = T_n$

III $u_n(x,t) = E_n \sin nx \cos nt \qquad (E_n = A_n D_n)$

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin nx \cos nt$$

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin nx \qquad = k \sin x - \tfrac{1}{2} \sin 2x$$

$$\Rightarrow \begin{cases} E_1 = k \\ E_2 = -\tfrac{1}{2} \\ E_n = 0, \quad n \geq 3 \end{cases}$$

$$u(x,t) = \underline{\underline{k \sin x \cos t - \tfrac{1}{2} \sin 2x \cos 2t}}$$

$$\sin\alpha \cos\beta = \tfrac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

$$\Rightarrow \quad \sin x \cos t = \tfrac{1}{2}\left(\sin(x+t) + \sin(x-t)\right)$$
$$\sin 2x \cos 2t = \tfrac{1}{2}\left(\sin 2(x+t) + \sin 2(x-t)\right)$$

$$\begin{aligned} u(x,t) &= \tfrac{k}{2}\left(\sin(x+t) + \sin(x-t)\right) - \tfrac{1}{2}\cdot\tfrac{1}{2}\left(\sin 2(x+t) + \sin 2(x-t)\right) \\ &= \tfrac{k}{2} \sin(x+t) - \tfrac{1}{4} \sin 2(x+t) + \tfrac{k}{2} \sin(x-t) - \tfrac{1}{4} \sin 2(x-t) \\ &= \tfrac{1}{2}\left(k \sin(x+t) - \tfrac{1}{2} \sin 2(x+t)\right) + \tfrac{1}{2}\left(k \sin(x-t) - \tfrac{1}{2} \sin 2(x-t)\right) \\ &= \tfrac{1}{2}\left(f(x+t) + f(x-t)\right), \quad \text{missä } f(y) = k \sin y - \tfrac{1}{2} \sin 2y \end{aligned}$$

(= alkupoikkeama)

IV
$$\begin{cases} u(x,0) = x(L-x) = Lx - x^2 = f(x) \\ u_t(x,0) = 0 \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

$$u_{tt} = c^2 u_{xx} \qquad (c \text{ vakio})$$

yrite: $u = X(x)T(t)$

$$XT'' = c^2 X'' T \qquad \| : XTc^2$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -p^2$$

1° $X'' = -p^2 X$

$\Rightarrow X = A\cos px + B\sin px$

$$u(0,t) = X(0)\,T(t) = 0$$
$$X(0) = 0$$
$$A \cdot 1 + B \cdot 0 = 0$$
$$A = 0$$

$$u(L,t) = X(L)\,T(t) = 0$$
$$X(L) = 0$$
$$B \sin pL = 0$$

$B = 0$ tai $\sin pL = 0$

ei mielekäs

$$pL = n\pi \quad , n \in \mathbb{N}$$
$$p = n\pi / L$$

$$\Rightarrow X_n = B_n \sin \frac{n\pi x}{L}$$

2° $T'' = -p^2 c^2 T$

$\Rightarrow T = C\cos pct + D\sin pct$

$T' = -Cpc\sin pct + Dpc\cos pct$

$$u_t(x,0) = X(x) = T'(0) = 0$$
$$T'(0) = 0$$

IV

$$D \rho c \cdot 1 = 0$$
$$D = 0$$

$$\Rightarrow T_n = C_n \cos \frac{n\pi c t}{L}$$

$$u_n = X_n T_n = B_n \sin \frac{n\pi x}{L} C_n \cos \frac{n\pi c t}{L}$$
$$= E_n \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L}$$

$$u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L}$$

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{L} = f(x)$$

Muodostetaan f:n pariton jatke $\tilde{f}$; $\tilde{f}(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

$$\tilde{f}(x) = \begin{cases} f(x) & , x > 0 \\ -f(-x) & , x < 0 \end{cases} \qquad (a_n = 0 \quad \forall\, n \in \mathbb{N} \cup \{0\})$$
$$= \begin{cases} xL - x^2 & , x > 0 \\ xL + x^2 & , x < 0 \end{cases}$$

$$\sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{L} = f(x) = \tilde{f}(x) \qquad (x > 0)$$
$$\sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$\Rightarrow E_n = b_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \sin \frac{n\pi x}{L} dx$$
$$= \frac{1}{L} \int_{-L}^{0} (xL + x^2) \sin \frac{n\pi x}{L} dx + \frac{1}{L} \int_{0}^{L} (xL - x^2) \sin \frac{n\pi x}{L} dx$$
$$= \ldots$$
$$= \frac{4L^2}{n^3 \pi^3} \underbrace{(1 - \cos n\pi)}_{\substack{0,\ n \text{ parillinen} \\ 2,\ n \text{ pariton}}}$$

IV $u(x,t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{4L^2}{n^3\pi^3}(1-\cos n\pi) \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L}$

$$= \frac{8L^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L}$$

V tehtävästä 3: $\sin \frac{n\pi x}{L} \cos \frac{nc\pi t}{L} = \frac{1}{2}\left(\sin \frac{n\pi}{L}(x+ct) + \sin \frac{n\pi}{L}(x-ct)\right)$

$$u(x,t) = \frac{8L^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3} \frac{1}{2}\left(\sin \frac{n\pi}{L}(x+ct) + \sin \frac{n\pi}{L}(x-ct)\right)$$

$$= \frac{1}{2} \cdot \frac{8L^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{L}(x+ct) + \frac{1}{2} \frac{8L^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{L}(x-ct)$$

$$= \frac{1}{2} f^*(x+ct) + \frac{1}{2} f^*(x-ct) \quad , \text{ missä}$$

$$f^*(x) = \frac{8L^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}$$

$$\sim u(x,0) = f(x)$$

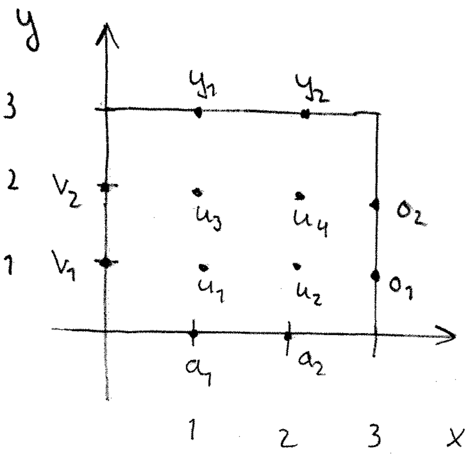
VI

(a)

$$u_{xx}(x,y) \approx \frac{1}{h^2}\left[u(x-h,y) - 2u(x,y) + u(x+h,y)\right]$$

$$+ \quad u_{yy}(x,y) \approx \frac{1}{h^2}\left[u(x,y-h) - 2u(x,y) + u(x,y+h)\right]$$

$$\Delta u(x,y) \approx \frac{1}{h^2}\left[u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y)\right] = 0 \quad (*)$$



← u:n arvot diskretointipisteissä

Kirjoitetaan diskretoitu yhtälö (*) pisteissä (1,1), (1,2), (2,1) ja (2,2):

$(1,1): \quad v_1 + u_2 + a_1 + u_3 - 4u_1 = 0$

$(1,2): \quad u_1 + o_1 + a_2 + u_4 - 4u_2 = 0$

$(2,1): \quad v_2 + u_4 + u_1 + y_1 - 4u_3 = 0$

$(2,2): \quad u_3 + o_2 + u_2 + y_2 - 4u_4 = 0$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -v_1 - a_1 \\ -o_1 - a_2 \\ -v_2 - y_1 \\ -o_2 - y_2 \end{bmatrix}$$

(b) $h = 1 \quad v_1 = v_2 = 0 \quad a_1 = 1^3 = 1 \quad a_2 = 2^3 = 8 \quad o_1 = 27 - 9 \cdot 1^2 = 18 \quad o_2 = 27 - 9 \cdot 2^2 = -9$

$y_1 = 1^3 - 27 \cdot 1 = -26 \quad y_2 = 2^3 - 27 \cdot 2 = -46$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \vec{u} = \begin{bmatrix} -1 \\ -26 \\ 26 \\ 55 \end{bmatrix} \Rightarrow \vec{u} = \begin{bmatrix} -2 \\ 2 \\ -11 \\ -16 \end{bmatrix}$$