LV 10 tehtävä I

$$\begin{cases} x' = 200x - 4xy \\ y' = -150y + 2xy \end{cases}$$

$x(t)$ = jänisten lkm

$y(t)$ = kettujen lkm

Eulerin menetelmä: $z_{j+1} = z_j + h\, f(t_j, z_j)$

$h = 0{,}2$

$t_j = jh, \quad j = 0, 1, 2 ..$

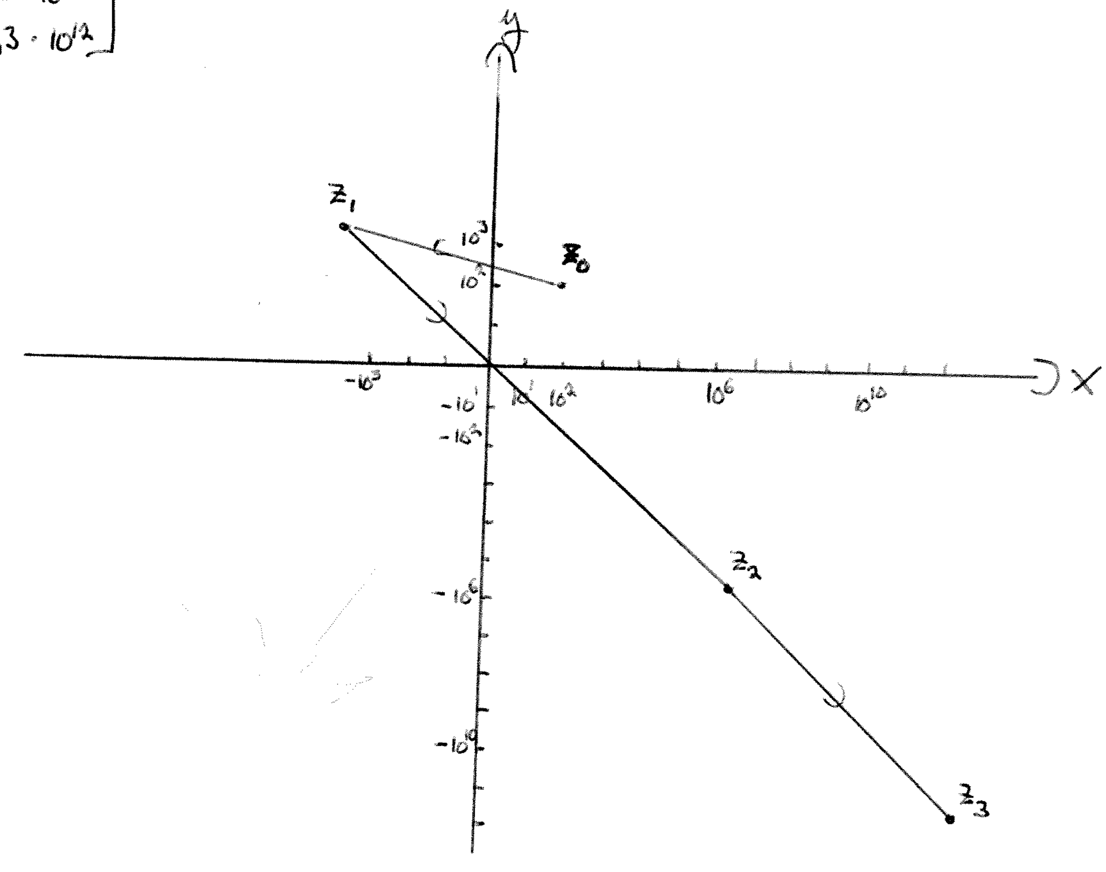
$z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad z_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$

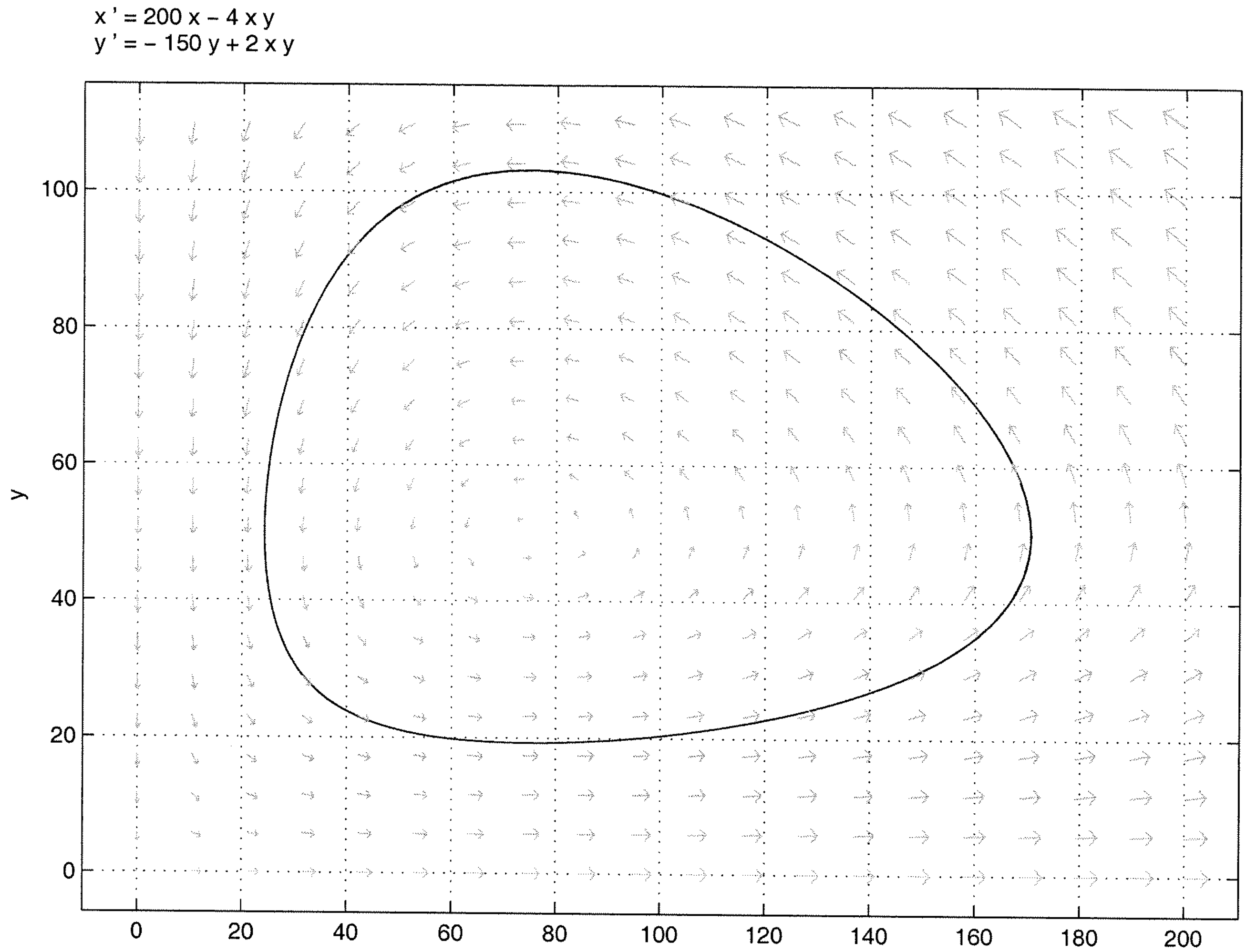
$f(z) = \begin{bmatrix} 200x - 4xy \\ -150y + 2xy \end{bmatrix}$

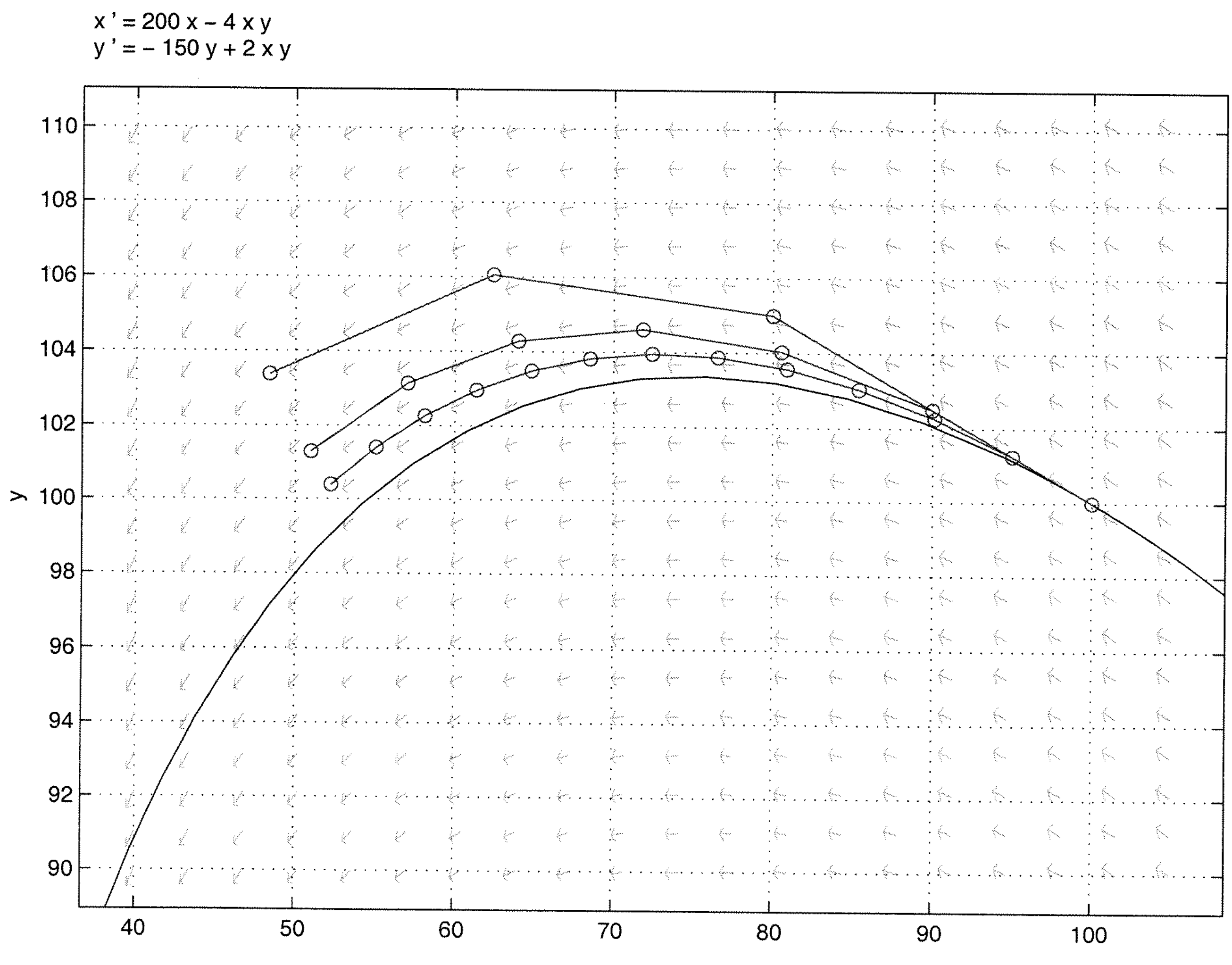
$$z_1 = z_0 + h\, f(z_0)$$
$$= \begin{bmatrix} 100 \\ 100 \end{bmatrix} + 0{,}2 \begin{bmatrix} 200 \cdot 100 - 4 \cdot 100 \cdot 100 \\ -150 \cdot 100 + 2 \cdot 100 \cdot 100 \end{bmatrix} = \begin{bmatrix} -3900 \\ 1100 \end{bmatrix}$$

$$z_2 = z_1 + h f(z_1)$$
$$= \begin{bmatrix} -3\,900 \\ 1\,100 \end{bmatrix} + 0{,}2 \begin{bmatrix} 200 \cdot (-3900) - 4(-3900) \cdot 1100 \\ -150 \cdot 1100 + 2 \cdot (-3900) \cdot 1100 \end{bmatrix} = \begin{bmatrix} 3\,272\,100 \\ -1\,747\,900 \end{bmatrix}$$

$$z_3 = z_2 + h\, f(z_2)$$
$$= \begin{bmatrix} 3\,272\,100 \\ -1\,747\,900 \end{bmatrix} + 0{,}2 \begin{bmatrix} 200 \cdot 3\,272\,100 - 4 \cdot 3\,272\,100 \cdot (-1\,747\,900) \\ -150 \cdot (-1\,747\,900) + 2 \cdot 3\,272\,100 \cdot (-1\,747\,900) \end{bmatrix}$$
$$\approx \begin{bmatrix} 4{,}5 \cdot 10^{12} \\ -2{,}3 \cdot 10^{12} \end{bmatrix}$$




x ' = 200 x − 4 x y
y ' = − 150 y + 2 x y
y
0
20
40
60
80
100
0
20
40
60
80
100
120
140
160
180
200


x ' = 200 x − 4 x y
y ' = − 150 y + 2 x y
y
90
92
94
96
98
100
102
104
106
108
110
40
50
60
70
80
90
100

LV10 Tehtävä 1

Sama järkevämmillä askelpituuksilla, Matlabilla:

```
>> h = 10^(-3); z=[100; 100]; zlist=[z];
>> for f = 1 : 3;
     z = z + h * [200*z(1) - 4*z(1)*z(2); -150*z(2) + 2*z(1)*z(2)];
     zlist=[zlist z];
   end
>> zlist1=zlist

zlist1 =

  100.0000   80.0000   62.4000   48.4099
  100.0000  105.0000  106.0500  103.3775
```

Otetaan sitten puolet lyhempiä askelia mutta kaksi kertaa enemmän:

```
>> h = 10^(-3)/2; z=[100; 100]; zlist=[z];
>> for f = 1 : 3*2;
     z = z + h * [200*z(1) - 4*z(1)*z(2); -150*z(2) + 2*z(1)*z(2)];
     zlist=[zlist z];
   end
>> zlist2=zlist

zlist2 =

  100.0000   90.0000   80.5500   71.8446   63.9970   57.0489   50.9860
  100.0000  102.5000  104.0375  104.6149  104.2848  103.1374  101.2859

>> h = 10^(-3)/4; z=[100; 100]; zlist=[z];
>> for f = 1 : 3*4;
     z = z + h * [200*z(1) - 4*z(1)*z(2); -150*z(2) + 2*z(1)*z(2)];
     zlist=[zlist z];
   end
>> zlist4=zlist

zlist4 =

  Columns 1 through 7

  100.0000   95.0000   90.1312   85.4208   80.8904   76.5568   72.4321
  100.0000  101.2500  102.2625  103.0362  103.5730  103.8781  103.9589

  Columns 8 through 13

   68.5237   64.8354   61.3674   58.1172   55.0799   52.2489
  103.8255  103.4893  102.9633  102.2615  101.3982  100.3883

>> plot(zlist1(1,:), zlist1(2,:),'r-o')
>> plot(zlist2(1,:), zlist2(2,:),'r-o')
>> plot(zlist4(1,:), zlist4(2,:),'r-o')
```

II $\quad y' = Ay \quad , y(0) = y_0 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad , A = \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix}$

$\Rightarrow y = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

om. arvot $\quad \det(A - \lambda I) = 0$

$(3-\lambda)(-4-\lambda) + 10 = 0$

$\lambda^2 + \lambda - 12 + 10 = 0$

$\lambda^2 + \lambda - 2 = 0$

$\Rightarrow \lambda_1 = -2 \quad , \lambda_2 = 1$

om. vektorit $\quad (A - \lambda_1 I) x_1 = 0$

$\begin{bmatrix} 5 & -2 \\ 5 & -2 \end{bmatrix} x_1 = 0 \quad \Rightarrow x_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$(A - \lambda_2 I) x_2 = 0$

$\begin{bmatrix} 2 & -2 \\ 5 & -5 \end{bmatrix} x_2 = 0 \quad \Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a) $y(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2e^{-2t} + e^{t} \\ 5e^{-2t} + e^{t} \end{bmatrix}$

$y(0) = c_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$$\begin{cases} 2c_1 + c_2 = 3 \\ 5c_1 + c_2 = 6 \end{cases}$$
$-$

$-3c_1 = -3$

$c_1 = 1 \quad \Rightarrow c_2 = 3 - 2c_1$
$= 1$

$y(0{,}2) \approx \begin{bmatrix} 2{,}562 \\ 4{,}573 \end{bmatrix}$

b) Euler: $\quad y_{n+1} = y_n + h\, A y_n \quad , h = 0{,}1$

$y_1 = y_0 + 0{,}1\, A y_0$

$= \begin{bmatrix} 3 \\ 6 \end{bmatrix} + 0{,}1 \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2{,}7 \\ 5{,}1 \end{bmatrix}$

$y_2 = y_1 + 0{,}1\, A y_1$

$= \begin{bmatrix} 2{,}7 \\ 5{,}1 \end{bmatrix} + 0{,}1 \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 2{,}7 \\ 5{,}1 \end{bmatrix} = \begin{bmatrix} 2{,}49 \\ 4{,}41 \end{bmatrix} = y_E$

II c) Heun:
$$\begin{cases} y^*_{n+1} = y_n + hAy_n \\ k_n = \frac{1}{2}(Ay_n + Ay^*_{n+1}) \\ y_{n+1} = y_n + hk_n \end{cases} \qquad h = 0{,}2$$

$$y_1^* = y_0 + 0{,}2\,Ay_0$$
$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix} + 0{,}2 \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2{,}4 \\ 4{,}2 \end{bmatrix}$$

$$k_0 = \frac{1}{2}\left(\begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 2{,}4 \\ 4{,}2 \end{bmatrix} \right) = \begin{bmatrix} -2{,}1 \\ -6{,}9 \end{bmatrix}$$

$$y_1 = y_0 + 0{,}2\,k_0 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + 0{,}2 \begin{bmatrix} -2{,}1 \\ -6{,}9 \end{bmatrix} = \begin{bmatrix} 2{,}58 \\ 4{,}62 \end{bmatrix} = y_H$$

d)
$$\| y(0{,}2) - y_E \| = \sqrt{(2{,}562 - 2{,}49)^2 + (4{,}573 - 4{,}41)^2} \approx 0{,}178$$
$$\| y(0{,}2) - y_H \| = \sqrt{(2{,}58 - 2{,}562)^2 + (4{,}62 - 4{,}573)^2} \approx 0{,}05$$

$$\max\{ |2{,}562 - 2{,}49|, |4{,}573 - 4{,}41| \} = 0{,}163$$
$$\max\{ |2{,}58 - 2{,}562|, |4{,}62 - 4{,}573| \} = 0{,}047$$

Heun tarkempi molemmissa vertailuissa

III $y' = f(y) = ay$, $y(0) = y_0 = 1$

$$k_1 = f(y_0) = a$$
$$k_2 = f(y_0 + \tfrac{h}{2}k_1) = a(1 + \tfrac{h}{2}a) = a + \tfrac{1}{2}ha^2$$
$$k_3 = f(y_0 + \tfrac{h}{2}k_2) = a(1 + \tfrac{h}{2}(a + \tfrac{1}{2}ha^2)) = a + \tfrac{1}{2}ha^2 + \tfrac{1}{4}h^2a^3$$
$$k_4 = f(y_0 + hk_3) = a(1 + h(a + \tfrac{1}{2}ha^2 + \tfrac{1}{4}h^2a^3)) = a + ha^2 + \tfrac{1}{2}h^2a^3 + \tfrac{1}{4}h^3a^4$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1 + \frac{h}{6}(a + 2a + ha^2 + 2a + ha^2 + \tfrac{1}{2}h^2a^3 + a + ha^2 + \tfrac{1}{2}h^2a^3 + \tfrac{1}{4}h^3a^4)$$
$$= 1 + \frac{h}{6}(6a + 3ha^2 + h^2a^3 + \tfrac{1}{4}h^3a^4)$$
$$= 1 + ah + \tfrac{1}{2}h^2a^2 + \tfrac{1}{6}h^3a^3 + \tfrac{1}{24}h^4a^4$$

Vertaa eksponenttifunktion sarjakehitelmään (Taylorin sarjaan)
$$e^{ah} = \sum_{k=0}^{\infty} \frac{1}{k!}(ah)^k$$

y_1 = 5 ensimmäistä termiä ko. sarjasta