Mat-1.3345 Diff. yhtälöiden inversio-ongelmat
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MALLIT

1. Osoita Fourier-muunnokselle, että

$$\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$$

on jatkuva, ts. kaikilla $\alpha, \beta \in \mathbb{N}$ on olemassa $m, \tilde{\alpha}_j, \tilde{\beta}_j \in \mathbb{N}$, $j = 1, \dots, m$, $C_{\alpha,\beta} > 0$ s.e.

$$\|\mathcal{F}u\|_{\alpha,\beta} \le C_{\alpha,\beta} \left(\sum_{j=1}^{m} \|u\|_{\tilde{\alpha}_j, \tilde{\beta}_j} \right).$$

Määritelmän mukaan

$$u \in S(\mathbb{R}) \Leftrightarrow u \in C^\infty(\mathbb{R}),\ \forall \alpha, \beta \in \mathbb{N}: \|u\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta f(x)| < \infty$$

Lasketaan auki

$$\|\mathcal{F}u\|_{\alpha,\beta} = \sup_{\xi \in \mathbb{R}} \left| \xi^\alpha \partial_\xi^\beta \int_{\mathbb{R}} e^{-ix\xi} u(x)\,dx \right|$$

$$\overset{(LDCT)}{=} \sup \left| \xi^\alpha \int_{\mathbb{R}} (-ix)^\beta e^{-ix\xi} u(x)\,dx \right|$$

$$= \sup \left| \int_{\mathbb{R}} (-ix)^\beta \frac{1}{(-i)^\alpha} \partial_x^\alpha e^{-ix\xi} u(x)\,dx \right|$$

$$= \sup \left| \int_{\mathbb{R}} \left(\partial_x^\alpha e^{-ix\xi} \right) x^\beta u(x)\,dx \right|$$

$$= \sup \left| \int_{\mathbb{R}} e^{-ix\xi} \partial_x^\alpha \left(x^\beta u(x) \right) dx \right|$$

$$\overset{\alpha' = \min(\alpha,\beta)}{=} \sup \left| \int_{\mathbb{R}} e^{-ix\xi} \sum_{j=0}^{\alpha'} \binom{\alpha}{j} (\partial_x^j x^\beta)(\partial_x^{\alpha-j} u)(x)\,dx \right|$$

$$\le \sup \sum_{j=0}^{\alpha'} \binom{\alpha}{j} \frac{\beta!}{(\beta-j)!} \left| \int_{\mathbb{R}} e^{-ix\xi} x^{\beta-j} (\partial_x^{\alpha-j} u)(x)\,dx \right|$$

Jaetaan integraali kahteen osaan

$$I = \left|\int_{\mathbb{R}}\right| \le \left|\int_{[-1,1]}\right| + \left|\int_{\mathbb{R}\setminus[-1,1]}\right| = I_1 + I_2$$

Nyt $I_1 \le 2\,\|u\|_{\beta-j,\alpha-j}$ ja

$$I_2 = \left|\int_{\mathbb{R}\setminus[-1,1]} e^{-ix\xi}\, x^{-2}\, x^{\beta-j+2}(\partial_x^{\alpha-j}u)(x)\,dx\right|$$

$$\le \tilde{C}_{\alpha,\beta,j}\,\|u\|_{\beta-j+2,\alpha-j}$$

Näemme että väite pätee.

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2. Laske

a) $\mathcal{F}(\delta_0)$, $\delta_0 \in S'(\mathbb{R})$ b) $\mathcal{F}(H)$, $H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$

c) (Demotehtävä) $\mathcal{F}(\text{p.v. } \frac{1}{x})$

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a) $\delta_0 : f \mapsto f(0)$

δ_0 on selvästi lineaarinen.

Olet. $f_j \to f$ $S(\mathbb{R})$:ssä.

$$\lim_{j\to\infty} |\delta_0(f_j) - \delta_0(f)| = \lim |f_j(0) - f(0)|$$

$$\le \lim \|f_j - f\|_{0,0} = 0$$

$\Rightarrow \delta_0 \in S'(\mathbb{R})$

$$\langle \mathcal{F}(\delta_0), f\rangle = \langle \delta_0, \mathcal{F}f\rangle = (\mathcal{F}f)(0)$$

$$= \int_{\mathbb{R}} f(x)\,dx$$

$\Rightarrow \mathcal{F}(\delta_0) = \mathbb{1}(x)$, $\mathbb{1} : x \mapsto 1$.

b) Olkoon

$$H_\varepsilon(x) := \begin{cases} e^{-\varepsilon x} & , \ x \geq 0 \\ 0 & \ x < 0 \end{cases}$$

Selvästi $H_\varepsilon \xrightarrow[\varepsilon \to 0]{S'} H$. Myös $H_\varepsilon \in L^1(\mathbb{R})$, joten

$$(\mathcal{F}H_\varepsilon)(\xi) = \int_0^\infty e^{-(\varepsilon + i\xi)x}\, dx = \frac{1}{\varepsilon + i\xi}.$$

Määritellään temperoidut distribuutiot

$$(\xi \pm i0)^{-1} : \varphi \mapsto \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \frac{\varphi(\xi)}{\xi \pm i\varepsilon}\, d\xi.$$

Voidaan osoittaa $(\xi \pm i0)^{-1} \in S'(\mathbb{R})$.
Nyt

$$\lim_{\varepsilon \to 0} \langle \mathcal{F}H_\varepsilon, f \rangle = \lim \langle H_\varepsilon, \mathcal{F}f \rangle = \langle H, \mathcal{F}f \rangle = \langle \mathcal{F}H, f \rangle,$$

joten $\mathcal{F}H_\varepsilon \xrightarrow{S'(\mathbb{R})} \mathcal{F}H$ ja

$$(\mathcal{F}H) : f \mapsto \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \frac{f(\xi)}{\varepsilon + i\xi}\, d\xi$$

eli

$$\mathcal{F}H = -i(\xi - i0)^{-1}.$$

c) Osoitamme ensin

$$(\xi + i0)^{-1} + (\xi - i0)^{-1} = 2\,\text{p.v.}\,\frac{1}{\xi} \qquad (1)$$

$$I = \langle (\xi + i0)^{-1} + (\xi - i0)^{-1}, \varphi \rangle$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left(\frac{1}{\xi + i\varepsilon} + \frac{1}{\xi - i\varepsilon} \right) \varphi(\xi)\, d\xi$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \frac{2\xi}{\xi^2 + \varepsilon^2}\, \varphi(\xi)\, d\xi$$

φ voidaan hajoittaa muotoon

$$\varphi(\xi) = \varphi(0) + \xi\psi(\xi),$$

missä ψ on jatkuva. Jaetaan integraali kahteen osaan

$$I = \lim_{\varepsilon} \int_{\mathbb{R}} = \lim_{\varepsilon} \int_{\mathbb{R}\setminus[-1,1]} + \lim_{\varepsilon} \int_{[-1,1]} = I_1 + I_2.$$

$$I_2 = \lim_{\varepsilon} \int_{-1}^{1} \frac{2\xi}{\xi^2+\varepsilon^2}\left(\varphi(0) + \xi\psi(\xi)\right) d\xi$$

$$= \lim \int_{-1}^{1} \frac{2\xi^2}{\xi^2+\varepsilon^2}\psi(\xi)\, d\xi = \int_{-1}^{1} \psi(\xi)\, d\xi$$

Näin ollen

$$I = 2\int_{\mathbb{R}\setminus[-1,1]} \frac{\varphi(\xi)}{\xi}\, d\xi + 2\int_{-1}^{1} \psi(\xi)\, d\xi$$

$$= 2\langle \text{p.v.}\tfrac{1}{\xi}, \varphi\rangle$$

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(1) $\Rightarrow$

$$2\,\text{p.v.}\frac{1}{\xi} = i\left(-i(\xi+i0)^{-1} - i(\xi-i0)^{-1}\right)$$

$$= i\left(-\underset{x\to\xi}{F}\{H(-x)\} + FH\right)$$

$$= i\underset{x\to\xi}{F}\{H(x) - H(-x)\} = 2i\underset{x\to\xi}{F}\{\operatorname{sgn}(x)\}$$

Siis

$$F\{\text{p.v.}\tfrac{1}{\xi}\} = i\underset{\xi\to x}{F}\{\underset{x\to\xi}{F}\operatorname{sgn}(x)\} = 2\pi i \operatorname{sgn}(-\cdot)$$

$$= -2\pi i \operatorname{sgn}(\cdot) \quad S'(\mathbb{R}):\text{ssä}$$

3. a) Jos $x_0, y_0 \in \mathbb{R}$ niin

$$\delta_{x_0} \otimes \delta_{y_0} = \delta_{(x_0,y_0)} \in S'(\mathbb{R}^2)$$

$\delta_{x_0} \otimes \delta_{y_0}$ on se $S'(\mathbb{R}^2)$:n alkio, jolle pätee kaikilla funktioilla muotoa $f(x,y) = g(x)h(y)$, $g, h \in S'(\mathbb{R})$

$$\langle \delta_{x_0} \otimes \delta_{y_0}, f \rangle := g(x_0)h(y_0)$$

Tiedämme että joukko $A = sp\{f \mid f(x,y) = g(x)h(y),\ g, h \in S'(\mathbb{R})\}$ on tiheässä $S'(\mathbb{R}^2)$:ssa.

Olkoon $f \in S(\mathbb{R}^2)$ ja $\{f_n\} \subset A$ s.e. $f_n \to f$ $S(\mathbb{R}^2)$:ssa. Silloin

$$\langle \delta_{x_0} \otimes \delta_{y_0}, f \rangle = \lim_n \langle \delta_{x_0} \otimes \delta_{y_0}, f_n \rangle$$

$$= \lim_n f_n(x_0, y_0)$$

$$\overset{f_n \xrightarrow{S} f}{=} f(x_0, y_0)$$

$$= \langle \delta_{(x_0,y_0)}, f \rangle$$

$$\Rightarrow \delta_{x_0} \otimes \delta_{y_0} = \delta_{(x_0,y_0)} \quad S'(\mathbb{R}^2)\text{:ssa}$$

b) Olkoon $\varphi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \varphi(x)dx = 1$ ja $\varphi_j(x) := j\varphi(jx)$. Osoita

$$\lim_{j \to \infty} \varphi_j = \delta_0 \quad S'(\mathbb{R})\text{:ssä.}$$

$$\lim_j \langle \varphi_j, f \rangle = \lim_j \int_{\mathbb{R}} j\varphi(jx)f(x)dx$$

$$= \lim_j \int_{\mathbb{R}} \varphi(x) f(\tfrac{x}{j})\, dx$$

$$= f(0) \int_{\mathbb{R}} \varphi(x)dx = f(0)$$

4. Osoita a) $\frac{d^2}{dx^2}\left(\frac{1}{2}|x|\right) = \delta_0$ $S'(\mathbb{R})$:ssä

b) $\Delta\left(-\frac{1}{4\pi|x|}\right) = \delta_0$ $S'(\mathbb{R}^3)$:ssa

a) $$\left\langle \frac{d^2}{dx^2}\left(\frac{1}{2}|x|\right), f \right\rangle = \int_{\mathbb{R}} \frac{d^2}{dx^2}\left(\frac{1}{2}|x|\right) f(x)\, dx$$
$$= \int \frac{1}{2}|x| f''(x)\, dx$$
$$= 2\int_0^\infty \frac{1}{2} x f''(x)\, dx$$
$$= \Big/_0^\infty x f'(x) - \int_0^\infty f'(x)\, dx$$
$$= f(0) = \langle \delta_0, f \rangle$$

b) $$\left\langle \Delta\left(-\frac{1}{4\pi|x|}\right), f \right\rangle =$$

Huom. $\Delta\left(-\frac{1}{4\pi|x|}\right)$ ei ole integroituva $\mathbb{R}^3$:ssa

$$= \frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla \frac{1}{|x|} \cdot \nabla f(x)\, dx$$
$$= \frac{-1}{4\pi}\int_{\mathbb{R}^3} \frac{x}{|x|^3} \cdot \nabla f(x)\, dx$$
$$= \frac{-1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \partial_r f(r,\theta,\varphi) \sin\theta\, dr\, d\theta$$
$$= f(0)$$

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