INVERSE BOUNDARY SPECTRAL PROBLEMS

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Contents

Introduction

Many physical processes are described by partial differential equations and systems of partial differential equations. The coefficients of these equations, which may depend on spatial and time variables, describe the properties of the medium where these processes take place. Solutions of these equations are called fields. The so-called direct problem consists of finding these fields when we know applied sources, initial and boundary conditions, and, of course, parameters of the medium. However, in physics and practical applications, the problem under consideration is often somewhat opposite. It is the unknown properties of the medium, described by the coefficients of the differential equations, that are to be determined. For example, in geophysical explorations, one wants to know the density and the Lame parameters that describe the properties of the Earth. These parameters may be used to find oil and other mineral fields. To find these parameters, we can utilize the fact that they appear in the coefficients of equations of elasticity. Henceforth, elastic waves, which propagate in the Earth, depend upon these parameters. Therefore, there is a natural problem of finding elastic parameters from the measurements of elastic fields. This problem is called the inverse problem of elasticity. Similar problems appear when one wants to find electric permittivity and conductivity and magnetic permeability from the measurements of electromagnetic fields, or acoustic velocity from the sound measurements, etc. These examples show the importance of inverse problems both for understanding physical phenomena and practical applications.

Inverse problems do not often have a unique solution. The data obtained from measurements may be the same for different physical models. For this reason, the goal of inverse problems is to find all equivalent models, i.e., those models that correspond to equal measurements. In spite of the non-uniqueness, the situation in real life is not as bad as it sounds. Measured data together with some additional *a priori* information about the model may be sufficient to find the unique model that describes the process. Indeed, in many cases, the non-uniqueness is due only to a natural freedom in describing the process. For instance, the domain filled with the medium may be described in different coordinate systems. Clearly, the resulting solution, which actually describes the same process, will be written in different forms. Therefore, in solving inverse problems, we need to take this into account and find their solutions in a coordinate invariant form. On the other hand, sometimes there are natural coordinates relevant to a particular inverse problem. These coordinates are often the travel time coordinates as, for example, in the described geophysical inverse problem. These travel time coordinates determine the natural distances between points. The distance function gives rise to a Riemannian structure, so it is natural to consider the domain as a Riemannian manifold. Therefore, the reconstruction of the Riemannian manifold and the corresponding metric is an essential part of solving inverse problems. This reconstruction of the manifold is a generalization of the reconstruction of the parameters of the medium in the travel time coordinates.

As seen from the above description, inverse problems have a wide range of practical applications and their mathematical theory is based upon an interaction of analysis and geometry with some additional algebraic ideas. Therefore, this book is written mainly for analysts, applied mathematicians and other applied scientists and engineers having an interest in the mathematical foundations of inverse problems. We believe that this book is also useful for geometricians who are interested in the interdisciplinary research.

It is clear that a large variety of physical phenomena described earlier give rise to numerous different types of mathematical inverse problems. To describe this variety and to understand the special nature of the problems that are described in this book, we will provide a rather sketchy classification of mathematical inverse problems. Our classification is based on the difference in various types of mathematical models that describe the corresponding physical processes, as well as various types of measured data used to reconstruct the unknown parameters of the medium. Before providing this classification, we note that the mathematical theory of inverse problems is still very far from its completion. As yet, there are no exact solutions to a number of practically important inverse problems. To solve such problems, mathematicians use various approximation methods. Moreover, in practice, given data is usually incomplete and errorprone. This also makes it necessary to utilize approximations and numerical techniques to study inverse problems. Corresponding methods are described in a number of books, e.g., [EnHaNu], [G], [BkGo]. The main aim of this book is to develop a rigorous theory to solve several types of inverse problems exactly, rather than to discuss applied and numerical aspects of these problems. However, we believe that this book will be useful for practitioners. This is not only because the book contains a new invariant approach to solve inverse problems, but also because of the algorithmic nature of the methods developed. This means that one of the main features of the book is to provide some procedures to reconstruct unknown parameters. Similarly, these procedures can be used to construct numerical solutions of inverse problems.

In the next few paragraphs we will give a classification of inverse problems and discuss briefly those problems that will be considered later in the book. First, we distinguish between the one-dimensional, i.e., with one spatial variable, and the multidimensional inverse problems. The one-dimensional case has many specific properties that do not exist in the multidimensional one. Historically, the study of mathematical inverse problems has been started with the one-dimensional case. Works of a number of outstanding mathematicians ([Bo], [Lv], [GeLe], [Ma1]-[Ma4], [Kr1]-[Kr4]) in the 1940s and '50s resulted in thorough understanding of these problems and provided several powerful methods to solve them. These methods were further developed later (see, for example, such classical books as [Ma4], [ChSa], [PoTr], [BeBl2], etc).

This book deals mainly with the multidimensional inverse problems, so that our further classification and references to literature will deal only with the multidimensional case. However, the methods developed in this book are perfectly applicable to one-dimensional inverse problems. Since these methods are easier to understand in the one-dimensional case, we provide the one-dimensional variant of the method in Chapter 1. We believe that this is a good introduction to study inverse problems in the multidimensional case.

Physical processes in a medium are described by various fields, e.g., electromagnetic fields, elastic fields, etc. These fields are often of vectorial nature. This means that the corresponding equations are vector equations, or, in other words, systems of equations. However, sometimes, the physical nature of the process makes it possible to reduce the vector equations to scalar ones. This occurs, for example, in gravitation or acoustics. Clearly, the vector case is significantly more difficult to analyze and the mathematical theory of the vector inverse problems is substantially less developed (see, however, textbooks [CoKs], [Ki], [Ro1] and historical remarks to Chapter **??**. In this book we will consider only the scalar inverse problems, although some of the developed methods can be extended to the vector case.

As mentioned, our classification of inverse problems is based on differentiation between various types of mathematical models that give rise to the corresponding inverse problems, e.g., one-dimensional versus multidimensional, scalar versus vector, and also differentiation among various types of inverse data.

In general, data used in inverse problems contains some given or measured information about the corresponding fields. Usually, this information describes the behaviour of these fields on the boundary of the domain, occupied by the medium, or at infinity. In the first case, we speak about inverse boundary problems, and in the second, about inverse scattering problems. Inverse scattering problems are of great importance due to their role in quantum mechanics and gravitational theory, and, historically, are the first inverse problems where mathematically rigorous results were obtained (see, for example, [CoKs], [Nw], [Fa3], [LxPh], [Me]). Inverse boundary problems, which have a wide range of important applications to geosciences, medical imaging, non-destructive testing, process monitoring, etc., are quite different from inverse scattering problems and require rather different methods and techniques. This book is devoted to inverse boundary problems.

According to the type of time dependence, physical processes are divided into stationary and non-stationary ones. Therefore, the corresponding inverse problems are also divided into stationary and non-stationary ones. Non-stationary inverse problems are also called dynamical or evolutionary problems. Depending on the hyperbolic or parabolic nature of the fields, the dynamical inverse problems are subdivided into hyperbolic or parabolic ones (see, e.g., [Ki], [Ro1], [Is1], [Sh1]). Stationary inverse boundary problems are subdivided into inverse boundary spectral problems and fixed frequency inverse problems. Fixed frequency inverse problems use boundary measurements at a finite number of frequencies. Moreover, the vast majority of results for these problems are obtained when the frequency is equal to 0, i.e., for the static case. These problems go back to the famous paper by Calderon [Cl] with fundamental mathematical theory developed in [SyU], [NvKh] with the crucial role of Faddeev's Green function [Fa1]. During the last two decades, this theory has been significantly extended and developed, see e.g., [Na1], [Na2], [NkU1], [OPaS], and Notes in Chapter **??**. A good exposition of the fixed frequency inverse problems is given in [Is1].

The main goal of this book is to study the inverse boundary spectral problems. However, the approach to inverse problems, which will be developed, is a dynamical approach. It is based on the consideration of the corresponding wave equations and involves various techniques to study an initial-boundary value problem for the wave equation. Henceforth, many results that are obtained for the inverse boundary spectral problems are valid for the dynamical inverse boundary problems as described in Chapter **??**.

Next we will describe, without going into details, the principal features and ingredients of the approach developed in the book.

1. Control theory for hyperbolic partial differential equations.
2. Geometry of geodesics and metric properties of Riemannian manifolds.
3. Asymptotic solutions of hyperbolic partial differential equations and, in particular, Gaussian beams.
4. Coordinate and gauge invariance of inverse problems and corresponding groups of transformations.

In our approach, we combine these ideas whose history can be briefly described as follows:

1. The importance of control theory for inverse problems was first understood by Belishev [Be1]. He used control theory to develop the first variant of the boundary control (BC) method, which is the analytical backbone of the book (see [Be2] for recent developments).

2. Later, the ideas based on control theory were combined with the geometrical ones. The importance of geometry for inverse problems follows the fact that any elliptic second-order differential operator gives rise to a Riemannian metric in the corresponding domain. The role of this metric becomes clearer if we consider the solutions of the corresponding wave equation. Indeed, these waves propagate with the unit speed along geodesics of this Riemannian metric. These geometric ideas were introduced to the boundary control method in [BeKu3] and [Ku4].

3. The close relation between the wave equation and the corresponding Riemannian metric becomes particularly clear when we consider some special classes of asymptotic solutions of this equation, like Gaussian beams. This asymptotic solution of the wave equation has the form of a wave packet. It propagates like a particle along a geodesic of the Riemannian metric, which prompts the name "quasiphoton" for this solution. This property makes Gaussian beams a very convenient technique to solve dynamical inverse problems. This observation was made in [BeKa1] and [KaKu1].

4. In the study of inverse boundary problems for general elliptic differential operators, it is necessary to take into account their non-uniqueness. In fact, there is a group of transformations of the operator, which preserve the boundary spectral data. This group of transformations was first analyzed in [Ku1], [Ku2].

In this book, we study the inverse boundary spectral problem for a general self-adjoint second-order elliptic differential operator on a compact manifold with boundary. In particular, our considerations cover the case of elliptic operators in bounded domains in Euclidean spaces. Moreover, we show that the developed approach is applicable to the study of the dynamical inverse boundary problems for the corresponding wave and heat equations.

The book consists of two unequal parts. The first part, Chapter 1, is devoted to the one-dimensional inverse boundary spectral problem on a finite interval. In this part, we introduce the principal ideas and techniques of the approach. The one-dimensional inverse problems are very interesting by themselves and have numerous important applications. In connection with this, we give a self-contained exposition of the approach for this case. Those readers who are interested only in the one-dimensional inverse problems can read Chapter 1 only. In the one-dimensional case, these principal ideas become much simpler and clearer than in the multidimensional case. Therefore, the study of the one-dimensional case is very instructive for further development of the theory in the multidimensional case.

The main part of the book, Chapters **??**–**??**, are devoted to the multidimensional inverse boundary problem. In Chapter **??**, we will give a solution for the Gel'fand inverse boundary spectral problem for a general second order self-adjoint elliptic differential operator on a compact manifold with boundary. To this end, we will develop an approach to the multidimensional inverse problems based on the boundary control method. We will describe the group of gauge transformations of an operator that do not change the boundary spectral data. We will show that any orbit of this group contains a unique (Riemannian) Schrödinger operator. We will describe an algorithm to construct this Schrödinger operator and the underlying differential manifold from the boundary spectral data of any operator in the orbit.

Chapter **??** is devoted to various generalizations and extensions to other inverse boundary problems of the approach developed in Chapter **??**. There are four main subjects discussed in this chapter. First, we will analyze the relations between the inverse boundary spectral problems and the dynamical inverse boundary problems for the wave and heat equations. This will make it possible to reduce the dynamical inverse problems to the inverse boundary spectral ones. Second, we will show that it is possible to solve the dynamical inverse problem for the wave equation directly, without reducing it to the inverse boundary spectral problem. The corresponding method is analogous to the method developed in Chapter **??**. However, there is a significant difference. Namely, if we possess boundary data only on a finite interval of time, then we can reconstruct the operator only in a collar neighborhood of the boundary. The width of this collar neighborhood is determined by the metric generated by the operator. Third, we will consider the inverse boundary spectral problem with data given only on a part of the boundary. This kind of problem is often encountered in practice. Fourth, we will consider the inverse boundary spectral problems in bounded domains of Euclidean spaces. Using the results in Chapter **??**, we will describe the groups of transformations in this case. Furthermore, we will show that additional *a priori* information about the structure of operators makes the inverse problem uniquely solvable. This takes place, for example, for some isotropic operators. We also provide a reconstruction algorithm. Considerations in Chapter **??** are given in less detail than those in Chapter **??**.

Chapter **??** is of an auxiliary nature. Here we provide an analytical and geometrical background of the approach developed in Chapters **??** and **??**.

This book is mainly aimed at readers with a background in analysis. This determines the choice of material and the level of exposition in Chapter **??**. We expect our readers to know the basics of distribution theory, Sobolev spaces and partial differential equations. When presented material goes beyond the standard textbooks, we will prove the corresponding results. Otherwise, we will provide only the rigorous formulations of the statements with references to the literature in Notes at the end of the chapter. Chapter **??** contains a rather wide range of ideas and techniques from modern analysis and geometry. The authors believe that this diversity reflects the interdisciplinary nature of the mathematical theory of inverse problems and corresponds to the current state of their development.

Each chapter consists of several titled sections subdivided into subsections, which we still call sections. Numeration of formulae and statements, i.e., theorems, lemmas and corollaries, is unified throughout a chapter. In references to a formula, the first number refers to the chapter and the second to the numeration of this formula in the chapter. There are exercises in Chapters 1–**??**. They are not very difficult. Usually they refer to those parts of exposition, that can be proven by considerations similar to those that are given earlier in the text. There are also sections in Chapters **??** and **??** that are marked by ★. These sections may be skipped in the first reading. At the beginning of each chapter, there is a short description of its structure and contents. At the end of each chapter, there is also a section called Notes. It provides references to the results used in the chapter and also gives references to the relevant literature on the subjects covered in the chapter. This sections also contains some historical commentaries about the results discussed in the chapter.

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Chapter 1

One-dimensional inverse problem

In this chapter, we will study the one-dimensional inverse boundary spectral problem. In sections 1.1.1–1.1.3 we will give a precise formulation of this problem. In the rest of section 1.1 we will describe the admissible transformations that preserve the boundary spectral data and reduce the inverse problem for a general operator to that for a Schrödinger operator. In section 1.2, we will consider the initial-boundary value problem for the corresponding wave equation. In particular, we will obtain a formula for the Fourier coefficients of the waves in terms of the boundary spectral data. In section 1.3, we will describe the necessary controllability results for the wave equation and give a solution to the inverse problem for the Schrödinger operator. To do that, we will introduce the so-called slicing procedure. In section 1.4, we will describe the one-dimensional Gaussian beams. We will later apply them to obtain an alternative solution of the inverse problem.

1.1. Inverse problem and main result

1.1.1. In many applied sciences, there appear elliptic ordinary differential operators with real smooth coefficients,

$$\mathcal{A}y = -a(x)y''(x) + b(x)y'(x) + c(x)y(x), \quad a > 0, \tag{1.1}$$

for a function $y(x)$ with x varying in the interval $[0, l]$. For example, in modeling harmonic oscillations of an inhomogeneous string, we deal with the equation

$$\mathcal{A}y = -ay'' + by' + cy = \omega^2 y.$$

Here a, b and c are related to the density ρ, Hooke constant μ and stiffness f by the formulae

$$a = \mu\rho^{-1}, \quad b = -\mu'\rho^{-1}, \quad c = f,$$

and ω is the frequency of oscillations.

In the direct problems, we know the material parameters and our goal is to find $y(x)$. However, for many practical purposes, the goal is just the opposite. We have information about solutions $y(x)$ and would like to use this information to find the material parameters. Such problems are called inverse problems. In practice, information about the solutions $y(x)$ comes from the measurements, and different types of measurements give rise to different types of inverse problems. In this chapter, we will study the one-dimensional inverse boundary spectral problems related to operator (1.1).

1.1.2. To define an operator $\mathcal{A}$ of form (1.1) rigorously, it is necessary to make some assumptions on its coefficients and also functions $y(x)$. In this chapter, we assume that a, b, and c are real-valued, smooth, i.e., infinitely differentiable, functions on the closed interval $[0, l]$. The operator $\mathcal{A}$ is then defined on the functions $y \in H^2([0, l])$, which means that y, y' are continuous and $y'' \in L^2([0, l])$, which satisfies the Dirichlet boundary conditions

$$y|_{x=0} = y|_{x=l} = 0. \tag{1.2}$$

The functions satisfying all these conditions form the domain of the operator $\mathcal{A}$, which is denoted by $\mathcal{D}(\mathcal{A})$,

$$\mathcal{D}(\mathcal{A}) = \{y \in H^2([0, l]) : \; y(0) = y(l) = 0\}. \tag{1.3}$$

Any operator of form (1.1), which is often called a Sturm-Liouville operator, can be rewritten as

$$\mathcal{A}y = -m^{-1}g^{-1/2}(mg^{-1/2}y')' + cy \tag{1.4}$$

with some positive smooth functions $m(x)$ and $g(x)$.

Exercise 1.1 *Find $m(x)$ and $g(x)$ in terms of a and b.*

The function $m(x)$ determines a weighted space $L^2([0,l],dV)$,

$$dV = m dV_g = m g^{1/2} dx$$

with the inner product of the form

$$\langle y, z \rangle = \int_0^l y(x)\overline{z(x)} m(x) g^{1/2} dx. \tag{1.5}$$

Remark Representation (1.4) for the operator $\mathcal{A}$ and the volume elements dV and dV_g are taken in the form suitable for the multi-dimensional case. In particular, the metric, or length element, on the interval $[0,l]$ that corresponds to the operator $\mathcal{A}$ has the form

$$ds^2 = g(x)dx^2.$$

Integrating by parts, we can show that

$$\langle \mathcal{A}y, z \rangle = \langle y, \mathcal{A}z \rangle, \tag{1.6}$$

for any $y, z \in \mathcal{D}(\mathcal{A})$. This means that $\mathcal{A}$ is symmetric in $L^2([0,l],dV)$. Moreover, it is known that any operator $\mathcal{A}$ of form (1.4), (1.3) is self-adjoint in $L^2([0,l],dV)$. Its spectrum consists of the isolated eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, $\lambda_k \to +\infty$. The corresponding eigenfunctions $\varphi_k(x)$, $\varphi_k \in \mathcal{D}(\mathcal{A})$ satisfy

$$\mathcal{A}\varphi_k = \lambda_k \varphi_k,$$

or,

$$-a(x)\varphi_k''(x) + b(x)\varphi_k'(x) + c(x)\varphi_k(x) = \lambda_k \varphi_k(x), \quad x \in [0,l],$$

$$\varphi_k(0) = 0, \quad \varphi_k(l) = 0.$$

can be chosen to form an orthonormal basis in $L^2([0,l],dV)$, i.e., $\langle \varphi_k, \varphi_l \rangle = \delta_{kl}$.

1.1.3. We are now in the position to formulate the inverse boundary spectral problem for the operator $\mathcal{A}$.

Problem 1 *Let all eigenvalues, $\lambda_1, \lambda_2, \dots$, be known as well as the values at the boundary points $x = 0$ and $x = l$ of the scaled derivatives of the normalized eigenfunctions $g^{-1/2}(0)\varphi_1'(0)$, $g^{-1/2}(0)\varphi_2'(0), \dots$ and $g^{-1/2}(l)\varphi_1'(l)$, $g^{-1/2}(l)\varphi_2'(l), \dots$. Is it possible to find the coefficients m, g and c from these data?*

Problem 2 *Let all eigenvalues $\lambda_1, \lambda_2, \dots$, be known as well as the values at one boundary point, say $x = 0$, of the scaled derivatives of the normalized eigenfunctions $g^{-1/2}(0)\varphi_1'(0)$, $g^{-1/2}(0)\varphi_2'(0), \dots$. Is it possible to find the coefficients m, g and c from these data?*

The data used in the formulations of Problems 1 and 2 are called the boundary spectral data.

Definition 1.2 *The collection*

$$\{\lambda_j,\ g^{-1/2}(0)\varphi_j'(0),\ g^{-1/2}(l)\varphi_j'(l) :\ j = 1, 2, \dots\}$$

is called the boundary spectral data of operator $\mathcal{A}$ corresponding to Problem 1. The collection

$$\{\lambda_j,\ g^{-1/2}(0)\varphi_j'(0) :\ j = 1, 2, \dots\}$$

is called the boundary spectral data of operator $\mathcal{A}$ corresponding to Problem 2.

Remark The factor $g^{-1/2}$ in the formulation of the boundary spectral data appears due to the equation

$$\frac{dy}{ds} = g^{-1/2}\,\frac{dy}{dx}.$$

This makes the definition of the boundary spectral data suitable for the multidimensional case also.

In the following, we will concentrate on the more complicated Problem 2. However, in the multidimensional case, we will be mostly preoccupied with the multidimensional analog of Problem 1. [1]

[1] In the one-dimensional case, problems 1, 2 are practically equivalent. In fact, if we know the boundary data λ_n, $\alpha_n = g^{-1/2}(0)\varphi_n'(0)$, $n = 1, 2, \dots$, of Problem 1 and $\lambda_n \neq 0$ for all n, then $\beta_n = g^{-1/2}(l)\varphi_n'(l)$ can be found using formulae $\beta_n = C\alpha_n^{-1}\lambda_n \prod_{k\neq n}(1-\lambda_n/\lambda_k)^{-1}$. Due to the non-uniqueness of the inverse boundary spectral problem, constant C can be arbitrary positive (or negative) constant. The sign of the constant depends on the solution of an auxiliary problem.

1.1.4. As we are going to show, Problems 1 and 2 do not have a unique solution, m, g, and c. Indeed, we can change m, g and c, so that the boundary spectral data remains unchanged. To see this, let us consider two types of transformations: changes of coordinates and gauge transformations.

i) *Changes of coordinates.* Let $\widetilde{X}$ be a diffeomorphism, $\widetilde{X} : [0, l] \to [0, \widetilde{l}]$, i.e., $\widetilde{X} \in C^\infty([0, l])$, $\widetilde{X}' > 0$, $\widetilde{X}(0) = 0$, $\widetilde{X}(l) = \widetilde{l}$ with its inverse $\widetilde{X}^{-1}$ denoted by X. This diffeomorphism corresponds to the change of coordinates from x to $\widetilde{x} = \widetilde{X}(x)$. In the new coordinates $\widetilde{x}$, the function y becomes the function $\widetilde{y} = y \circ X$, i.e. $\widetilde{y}(\widetilde{x}) = y(X(\widetilde{x}))$. To preserve the inner product, we introduce new functions $\widetilde{m}$, $\widetilde{g}$,

$$\widetilde{m}(\widetilde{x}) = m(X(\widetilde{x})), \quad \widetilde{g}(\widetilde{x}) = g(X(\widetilde{x}))[X'(\widetilde{x})]^2. \tag{1.7}$$

Then

$$\langle y, z \rangle = \int_0^l y(x)\overline{z(x)}dV = \int_0^{\widetilde{l}} \widetilde{y}(\widetilde{x})\overline{\widetilde{z}(\widetilde{x})}d\widetilde{V} = \langle \widetilde{y}, \widetilde{z} \rangle, \tag{1.8}$$

where $d\widetilde{V} = \widetilde{m}\widetilde{g}^{1/2}d\widetilde{x}$. In these coordinates, the operator $\mathcal{A}$ becomes the operator $\widetilde{\mathcal{A}}$,

$$\widetilde{\mathcal{A}}\widetilde{y}(\widetilde{x}) = -\widetilde{m}^{-1}\widetilde{g}^{-1/2}(\widetilde{m}\widetilde{g}^{-1/2}\widetilde{y}')' + \widetilde{c}\widetilde{y}, \tag{1.9}$$

where $\widetilde{c} = c \circ X$. More precisely, this means that

$$(\mathcal{A}y)(X(\widetilde{x})) = \widetilde{\mathcal{A}}\widetilde{y}(\widetilde{x}).$$

In particular,

$$\widetilde{\mathcal{A}}\widetilde{\varphi}_k = \lambda_k \widetilde{\varphi}_k.$$

Clearly, $\widetilde{\varphi}_k$ satisfy the Dirichlet boundary conditions at $\widetilde{x} = 0$ and $\widetilde{x} = \widetilde{l}$ and, henceforth, $\widetilde{\varphi}_k$ are the eigenfunctions of operator $\widetilde{\mathcal{A}}$. Due to identity (1.8), $\widetilde{\varphi}_k$, $k = 1, 2, \dots$, remain orthonormalized, i.e. $\langle \widetilde{\varphi}_k, \widetilde{\varphi}_l \rangle = \delta_{kl}$. Moreover,

$$\widetilde{g}^{-1/2}(0)\widetilde{\varphi}_k'(0) = g^{-1/2}(0)\varphi_k'(0), \quad \widetilde{g}^{-1/2}(\widetilde{l})\widetilde{\varphi}_k'(\widetilde{l}) = g^{-1/2}(l)\varphi_k'(l),$$

for all $k = 1, 2, \dots$. This means that the boundary spectral data of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are the same.

1.1.5. *ii) Gauge transformations.* Let κ be a smooth positive function, $\kappa \in C^\infty([0,l])$, $\kappa > 0$. This function determines a transformation $\mathcal{A}_\kappa$ of the operator $\mathcal{A}$, which is called a gauge transformation, of $\mathcal{A}$

$$\mathcal{A}_\kappa u = \kappa \mathcal{A}(\kappa^{-1} u). \tag{1.10}$$

Operator $\mathcal{A}_\kappa$ has form (1.4) with m, g and c replaced by m_κ, g_κ, and c_κ,

$$m_\kappa = \kappa^{-2} m, \quad g_\kappa = g, \quad c_\kappa = \mathcal{A}(\mathbf{1}) \tag{1.11}$$

where $\mathbf{1}$ is the constant function $\mathbf{1}(x) = 1$.

Exercise 1.3 *Prove equations (1.11).*

Then the functions ψ_k,

$$\psi_k = \kappa \varphi_k, \quad k = 1, 2, \dots, \tag{1.12}$$

where φ_k are the eigenfunctions of $\mathcal{A}$, satisfy the equations

$$\mathcal{A}_\kappa \psi_k = \lambda_k \psi_k, \quad k = 1, 2, \dots,$$

as well as the Dirichlet boundary conditions. This means that λ_k and ψ_k are the eigenvalues and eigenfunctions of the operator $\mathcal{A}_\kappa$, which is self-adjoint in $L^2([0,l], dV_\kappa)$, where $dV_\kappa = m_\kappa g_\kappa^{1/2} dx$. Moreover, the functions ψ_k form an orthonormal basis in the space $L^2([0,l], dV_\kappa)$. Henceforth, the boundary spectral data of $\mathcal{A}_\kappa$ are given by λ_k and

$$g_\kappa^{-1/2}(0)\psi_k'(0) = \kappa(0)[g^{-1/2}(0)\varphi_k'(0)], \tag{1.13}$$

$$g_\kappa^{-1/2}(l)\psi_k'(l) = \kappa(l)[g^{-1/2}(l)\varphi_k'(l)],$$

where $k = 1, 2, \dots$ If $\kappa(0) = \kappa(l) = 1$, then the boundary spectral data of $\mathcal{A}$ and $\mathcal{A}_\kappa$, used in Problem 1, are the same and, if $\kappa(0) = 1$, then the boundary spectral data of $\mathcal{A}$ and $\mathcal{A}_\kappa$ used in Problem 2 are the same.

Summarizing, we see that any function $\kappa \in C^\infty([0,l])$, $\kappa > 0$ determines a transformation of an operator $\mathcal{A}$ that given by formula (1.10). This transformation is called the gauge transformation corresponding to κ.

When κ satisfies additional boundary conditions $\kappa(0) = \kappa(l) = 1$ or $\kappa(0) = 1$, we call the corresponding gauge transformation the normalized gauge transformation related to Problems 1 or 2.

1.1.6. In the previous section, we described two types of transformations of an operator $\mathcal{A}$. Next, we use them to make $\mathcal{A}$ as simple as possible. Our goal is to obtain a Schrödinger operator,

$$\mathcal{A}_0 \equiv -\frac{d^2}{dx^2} + q(x). \tag{1.14}$$

Lemma 1.4 *For any operator $\mathcal{A}$ of form (1.4) there exists a gauge transformation, followed by a change of coordinates that transform $\mathcal{A}$ into a Schrödinger operator*

$$\mathcal{A}_0 = -\frac{d^2}{d\widetilde{x}^2} + q(\widetilde{x})$$

on the interval $[0, \widetilde{l}]$. For this end, we should take

$$\kappa(x) = m^{1/2}(x), \tag{1.15}$$

$$\widetilde{x} = \widetilde{X}(x) = \int_0^x g^{1/2}(x')dx', \tag{1.16}$$

so that

$$\widetilde{l} = \widetilde{X}(l).$$

Proof. According to formula (1.11), gauge transformation (1.15) makes $m_\kappa = 1$, $g_\kappa = g$. According to formulae (1.7) and (1.9), the change of coordinates (1.16) keeps $\widetilde{m} = 1$ and makes $\widetilde{g} = 1$. □

Exercise 1.5 *Find the potential q in terms of m, g and c.*

We note that we can change the order of the transformations, making first the change of coordinates (1.16) followed by the gauge transformation which corresponds to $\widetilde{\kappa}(\widetilde{x}) = \kappa(X(\widetilde{x}))$.

1.1.7. Later, we will describe a method of the reconstruction of the potential q of a Schrödinger operator from the boundary spectral data. Lemma 1.4 shows that any operator of form (1.4), (1.3) can be transformed into a Schrödinger operator. Unfortunately, gauge transformations (1.15) are not, in general, normalized. This means that the boundary spectral data of the Schrödinger operator $\mathcal{A}_0$, which is used for the reconstruction of q, differs from the given boundary spectral data of the original operator $\mathcal{A}$ of form (1.4).

However, there is a method to find the boundary values of m from the boundary spectral data of $\mathcal{A}$. To this end, we use the asymptotics of the eigenvalues λ_k and eigenfunctions $\widetilde{\varphi}_k$ of the Schrödinger operator $\mathcal{A}_0$ which corresponds to the operator $\mathcal{A}$, (see Notes at the end of the chapter)

$$\lambda_k = \left(\frac{\pi k}{\widetilde{l}}\right)^2 + O(1),$$

$$\widetilde{\varphi}_k(\widetilde{x}) = \sqrt{\frac{2}{\widetilde{l}}}\sin\left(\frac{k\pi\widetilde{x}}{\widetilde{l}}\right) + O(k^{-1}), \tag{1.17}$$

$$\widetilde{\varphi}_k'(\widetilde{x}) = \sqrt{\frac{2}{\widetilde{l}}}\frac{k\pi}{\widetilde{l}}\cos\left(\frac{k\pi\widetilde{x}}{\widetilde{l}}\right) + O(k^{-1}).$$

Using the transformations described in Lemma 1.4, we see that

$$\widetilde{\varphi}_k(\widetilde{X}(x)) = m^{1/2}(x)\varphi_k(x).$$

Therefore, the boundary spectral data for the operator $\mathcal{A}$ has the form

$$\widetilde{\varphi}_k'(0) = m^{1/2}(0)[g^{-1/2}(0)\varphi_k'(0)], \tag{1.18}$$

$$\widetilde{\varphi}_k'(\widetilde{l}) = m^{1/2}(l)[g^{-1/2}(l)\varphi_k'(l)].$$

These formulae together with formulae (1.17) show that

$$m(0) = \lim_{k\to\infty}\frac{2\lambda_k^{3/2}}{\pi k[g^{-1/2}(0)\varphi_k'(0)]^2},$$

$$m(l) = \lim_{k\to\infty}\frac{2\lambda_k^{3/2}}{\pi k[g^{-1/2}(l)\varphi_k'(l)]^2}.$$

Combining these formulae with the equations (1.18), we obtain the boundary spectral data for the Schrödinger operator $\mathcal{A}_0$, which corresponds to an arbitrary general operator $\mathcal{A}$.

Summarizing considerations of sections 1.1.6, 1.1.7, we see that we can reduce the inverse boundary spectral problem for a general operator to the inverse boundary spectral problem for the corresponding Schrödinger operator $\mathcal{A}_0$.

1.1.8. Now we are in the position to formulate our main result for the one-dimensional inverse boundary spectral Problems 1 and 2. The key result is the following theorem.

Theorem 1.6 *Assume that* $\{\lambda_1,\ \lambda_2, \dots, \varphi_1'(0),\ \varphi_2'(0), \dots\}$ *are the boundary spectral data at* $x = 0$ *of a Dirichlet Schrödinger operator,* $\mathcal{A}_0 = -\frac{d^2}{dx^2} + q$ *on an interval* $[0, l]$*. Then these data determine* l *and* $q(x)$ *uniquely.*

Theorem 1.6 gives an affirmative answer to Problem 2.

This theorem is proven in sections 1.2 and 1.3 below. In fact, we will describe a procedure how to find l and reconstruct q. Obviously, Theorem 1.6 also gives an answer to Problem 1 in the case of a Schrödinger operator.

In the case of a general operator of form (1.4), we will first construct the corresponding Schrödinger operator. The operator $\mathcal{A}$ lies in the class of operators that can be obtained from this Schrödinger operator $\mathcal{A}_0$ by a gauge transformation with $\kappa(0) = m^{1/2}(0)$, $\kappa(l) = m^{1/2}(l)$, followed by a change of coordinates. All operators of this class have the same boundary spectral data and, henceforth, are indistinguishable from the boundary data. Moreover, Theorem 1.6 implies the following theorem for the general operators.

1.1.9.

Theorem 1.7 *Two operators* $\mathcal{A}$ *and* $\mathcal{B}$ *of form (1.4), (1.3) have the same boundary spectral data, if and only if they can be obtained from one another by a change of coordinates and a normalized gauge transformation.*

Proof. i) The part "if" of the theorem is already proven in sections 1.1.4, 1.1.5.

ii) To prove the part "only if", we first note that, due to (1.11), $m_{\mathcal{A}}(0) = m_{\mathcal{B}}(0)$, where we have denoted by $m_{\mathcal{A}}$, $l_{\mathcal{A}}$, etc. the corresponding quantities for the operator $\mathcal{A}$ and by $m_{\mathcal{B}}$, $l_{\mathcal{B}}$, etc. the corresponding quantities for the operator $\mathcal{B}$. Hence, the boundary spectral data of the corresponding Schrödinger operators $\mathcal{A}_0$ and $\mathcal{B}_0$ are the same and, by Theorem 1.6, $\mathcal{A}_0 = \mathcal{B}_0$.

In view of the remark at the end of section 1.1.6, the operator $\mathcal{A}_0$ is obtained from $\mathcal{A}$ by the following consecutive transformations.

First we make a change of coordinates and obtain an operator $\widehat{\mathcal{A}}$. Then we make a gauge transformation corresponding to $\kappa_{\mathcal{A}}$ with $\kappa_{\mathcal{A}}(0) = m_{\mathcal{A}}^{1/2}(0)$ and obtain $\mathcal{A}_0$. Analogously, to obtain $\mathcal{B}_0$ from $\mathcal{B}$, we first make a change of coordinates and obtain an operator $\widehat{\mathcal{B}}$. Then we make a gauge transformation corresponding to $\kappa_{\mathcal{B}}$ with $\kappa_{\mathcal{B}}(0) = m_{\mathcal{B}}^{1/2}(0)$.

Since $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ are gauge equivalent to the same Schrödinger operator $\mathcal{A}_0 = \mathcal{B}_0$, they are still defined on the same interval $[0, l]$. We have

$$\mathcal{A}_0 = \kappa_{\mathcal{A}} \widehat{\mathcal{A}} \kappa_{\mathcal{A}}^{-1} = \kappa_{\mathcal{B}} \widehat{\mathcal{B}} \kappa_{\mathcal{B}}^{-1},$$

which yields that

$$\widehat{\mathcal{B}} = (\kappa_{\mathcal{B}}^{-1} \kappa_{\mathcal{A}}) \widehat{\mathcal{A}} (\kappa_{\mathcal{B}}^{-1} \kappa_{\mathcal{A}})^{-1} = \kappa \widehat{\mathcal{A}} \kappa^{-1}.$$

Since $m_{\mathcal{A}}(0) = m_{\mathcal{B}}(0)$, this is a normalized gauge transformation. Therefore, the operator $\mathcal{B}$ is obtained from $\mathcal{A}$ by change of coordinates, a normalized gauge transformation and one more change of coordinates. Finally, we observe that the resulting transformation can be obtained as a combination of one normalized gauge transformation and one change of coordinates. □

1.1.10. The previous discussion shows that there is no chance to reconstruct all coefficients of a general operator (1.4) from its boundary spectral data. However, in many cases that are important for applications, we have some additional information about the form of the operator $\mathcal{A}$. Sometimes, this information is sufficient for the unique reconstruction of $\mathcal{A}$, when the class of gauge equivalent operators contain only one operator having the specified form. An obvious example is the Schrödinger operator. Other examples are given by the operators

$$\mathcal{A} = -c^2(x) \frac{d^2}{dx^2}, \quad \mathcal{B} = -\frac{d}{dx} \mu(x) \frac{d}{dx},$$

which appear, e.g., in the study of the inverse problem for an inhomogeneous string.

Exercise 1.8 *Show that the boundary spectral data of the operators $\mathcal{A}$ and $\mathcal{B}$ determine c and μ uniquely.*

1.2. Wave equation

1.2.1. This book is devoted to the inverse boundary spectral problems. However, our approach to solving this problem is based on the properties of the corresponding wave equation. For an operator $\mathcal{A}$ of form (1.1) or (1.4), the corresponding wave equation is

$$\partial_t^2 w + \mathcal{A}w \equiv \partial_t^2 w - a(x)\partial_x^2 w + b(x)\partial_x w + c(x)w =$$

$$= \partial_t^2 w - m^{-1}g^{-1/2}\partial_x(mg^{-1/2}\partial_x w) + cw = 0, \quad x \in [0,l] \quad (1.19)$$

where $w = w(x,t)$. A well-known and important property of the wave equation is the finite velocity of the wave propagation. For equation (1.19), this velocity, $v(x)$ is given by formula,

$$v(x) = a^{1/2}(x) = g^{-1/2}(x). \quad (1.20)$$

The velocity v determines the travel time τ between any two points x_1 and x_2, $0 \leq x_1 \leq x_2 \leq l$,

$$\tau(x_1, x_2) = \int_{x_1}^{x_2} \frac{dx}{v(x)} = \int_{x_1}^{x_2} g^{1/2}(x)dx. \quad (1.21)$$

From the point of view of physics, $\tau(x_1, x_2)$ is the time necessary for a perturbation at the point x_1 to reach the point x_2. The travel time can be interpreted as the physically meaningful distance between points. We can use the travel time, $\tau(x) = \tau(0,x)$ from the boundary point 0 to a variable point x as a new coordinate. The coordinate transformation from x to $\tau(x)$ is exactly the change of coordinates, which transforms a general operator to a Schrödinger one. Using also gauge transformation (1.15), which has the physical meaning of re-scaling of the dependent variable, we transform wave equation (1.19) into the wave equation

$$\partial_t^2 u + \mathcal{A}_0 u \equiv \partial_t^2 u - \partial_\tau^2 u + qu = 0, \quad (1.22)$$

for the function $u(x,t) = \kappa(x)w(x,t)$.

1.2.2. The remaining part of Chapter 1 is mainly devoted to the consideration of Problem 2 for the Dirichlet Schrödinger operator

$\mathcal{A}_0$ of form (1.14), (1.3). Let us consider the initial-boundary value problem for equation (1.22),

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + q(x)u = F(x,t), \\ u|_{x=0} = 0, \quad u|_{x=l} = 0, \\ u|_{t=0} = \partial_t u|_{t=0} = 0, \end{cases} \tag{1.23}$$

with $F \in L^2([0,l] \times [0,T])$ for any $T > 0$.

We start our considerations with the case $F \in C_0^\infty([0,l] \times [0,T])$. It is known that, in this case, initial-boundary value problem (1.23) has a unique solution $u \in C^\infty([0,l] \times [0,T])$ and $u(x,t) = 0$ for $x > t$, when $F(x,t) = 0$ for $x > t$. The last property follows from the finite velocity of the wave propagation, which is equal to 1 for wave equation (1.22) (see Notes in the end of the chapter). In the following, we need some less regular solutions. For this end, we prove estimates for the solution.

Lemma 1.9 *Let $u(x,t)$ be the solution of the initial-boundary value problem (1.23) for $F \in C_0^\infty([0,l] \times [0,T])$. Then*

$$H_u(t) \le C(T) \|F\|^2_{L^2([0,l]\times[0,T])}, \quad 0 \le t \le T, \tag{1.24}$$

$$|u(x,t)| \le C(T) \|F\|_{L^2([0,l]\times[0,T])}, \quad 0 \le x \le l, \quad 0 \le t \le T, \tag{1.25}$$

where

$$H_u(t) = \frac{1}{2} \int_0^l (|\partial_t u(x,t)|^2 + |\partial_x u(x,t)|^2 + |u(x,t)|^2) dx. \tag{1.26}$$

Proof. Using the wave equation in (1.23) and integration by parts, we obtain

$$\frac{dH_u}{dt} = \Re \int_0^l \Big(\partial_t u(x,t)\, \overline{\partial_t^2 u(x,t)} + \partial_t \partial_x u(x,t)\, \overline{\partial_x u(x,t)} +$$

$$+ \partial_t u(x,t)\, \overline{u(x,t)} \Big)\, dx =$$

$$= \Re \int_0^l \partial_t u(x,t) (\overline{F(x,t)} + (1 - q(x)) \overline{u(x,t)}) dx +$$

$$+ \Re (\partial_t u(l,t) \overline{\partial_x u(l,t)} - \partial_t u(0,t) \overline{\partial_x u(0,t)}).$$

The last term in this equation is equal to zero due to the boundary conditions in (1.23). Then,

$$\frac{dH_u}{dt} \leq \frac{1}{2}\|F(\cdot,t)\|^2 + (C_q+2)H_u, \tag{1.27}$$

where

$$C_q = \max_{x\in[0,l]} |q(x)|. \tag{1.28}$$

This inequality can be rewritten as

$$\frac{d}{dt}\left(e^{-(C_q+2)t}H_u\right) \leq \frac{1}{2}e^{-(C_q+2)t}\|F(\cdot,t)\|^2. \tag{1.29}$$

In view of the initial conditions in (1.23), $H_u(0)=0$, so that inequality (1.29) implies that

$$\begin{aligned} H_u(t) &\leq \frac{1}{2}\int_0^t e^{(C_q+2)(t-t')}\|F(\cdot,t')\|^2 dt' \qquad (1.30)\\ &\leq C(T)\|F\|^2_{L^2([0,l]\times[0,T])}. \end{aligned}$$

In particular,

$$\int_0^l |\partial_x u(x,t)|^2 dx \leq C(T)\|F\|^2_{L^2([0,l]\times[0,T])}, \tag{1.31}$$

so that, by the Cauchy inequality,

$$|u(x,t)| \leq \int_0^x |\partial_x u(x,t)|dx \leq C(T)\|F\|_{L^2([0,l]\times[0,T])}. \tag{1.32}$$

□

Estimate (1.25) makes it possible to define the solution of initial-boundary value problem (1.23) when $F \in L^2([0,l]\times[0,T])$. Indeed, let $F_n \to F$ in $L^2([0,l]\times[0,T])$ and $F_n \in C_0^\infty([0,l]\times[0,T])$. Denote by $u_n(x,t)$ the solution of initial-boundary value problem (1.23) with F replaced by F_n. Then, by inequality (1.25), there is a function $u(x,t) \in C([0,l]\times[0,T])$ such that

$$u_n(x,t) \to u(x,t), \quad 0 \leq x \leq l, \quad 0 \leq t \leq T.$$

This function $u(x,t)$ is called the solution of initial-boundary value problem (1.23). The previous considerations imply the following result

Corollary 1.10 *For $F \in L^2([0,l]\times[0,T])$ there is a unique solution $u \in C(([0,l]\times[0,T])$ for which estimates (1.24), (1.25) are valid. Moreover, if $F(x,t)=0$ for $x>t$ then $u(x,t)=0$ for $x>t$.*

Remark. It is known that, when $F \in L^2([0,l]\times[0,T])$, the function $u(x,t)$ coincides with the unique weak solution of initial-boundary value problem (1.23). Moreover, estimate (1.24) means that this weak solution belongs to the energy class. The properties and uniqueness of the weak solutions can be found in details from references given in Notes.

1.2.3. The initial-boundary value problem for the equation (1.22), which is related to the inverse boundary spectral problem, is the problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + q(x)u = 0, \\ u|_{x=0} = f(t), \quad u|_{x=l} = 0, \\ u|_{t=0} = \partial_t u|_{t=0} = 0, \end{cases} \tag{1.33}$$

for $f \in L^2([0,T])$. We denote $u(x,t) = u^f(x,t)$ to indicate the Dirichlet boundary value of u. Function f is also called the boundary source.

When $q=0$, the solution $u_0^f(x,t)$ of this problem, for $0<t<T$, is given by the formula

$$u_0^f(x,t) = \sum_{j=0}^{\infty}[f(t-x-2lj) - f(t+x-2l(j+1))], \tag{1.34}$$

when we continue $f(t)$ to be 0 for $t<0$. We point out that, for $t<T$, the sum in the right-hand side is finite. Indeed, all terms for $j>T/l$ vanish.

To construct the solution for the initial-boundary value problem (1.33), we consider the solution $u_1^f(x,t)$ of the following initial-boundary value problem,

$$\begin{cases} (\partial_t^2 - \partial_x^2 + q(x))u_1^f = -qu_0^f, \\ u_1^f|_{x=0} = 0, \quad u_1^f|_{x=l} = 0, \\ u_1^f|_{t=0} = \partial_t u_1^f|_{t=0} = 0. \end{cases} \tag{1.35}$$

By Corollary 1.10, $u_1^f \in C([0,l]\times[0,T])$. Then the unique solution of initial-boundary value problem (1.33) is given by the formula,

$$u^f(x,t) = u_0^f(x,t) + u_1^f(x,t). \tag{1.36}$$

Denote by $C([0,T],L^2([0,l]))$ the space of functions of t with values in $L^2([0,l])$, which depend continuously on t. We denote the norm of the space $L^2([0,l])$ by $||\cdot||$.

Lemma 1.11 *For any $f \in L^2([0,T])$, initial-boundary value problem (1.33) has a unique solution $u^f(x,t) \in C([0,T],L^2([0,l]))$, such that*

$$\|u^f(\cdot,t)\| \le C(T)\|f\|, \quad t < T.$$

Proof. Each term in the right-hand side of formula (1.34) is in the space $C([0,T],L^2([0,l]))$. Since the sum in (1.34) is finite, $u_0^f \in C([0,T],L^2([0,l]))$ and

$$\|u_0^f(\cdot,t)\| \le C(T)\|f\|.$$

Then the right-hand side in (1.35), i.e., the function $-qu_0^f \in L^2([0,l]\times[0,T])$. By Corollary 1.10, $u_1^f \in C([0,l]\times[0,T])$ and satisfies estimate (1.32) with $F = -qu_0^f$. Therefore,

$$|u_1^f(x,t)| \le C(T)\|qu_0^f\|_{L^2([0,l]\times[0,T])} \le C_1(T)\|f\|.$$

□

1.2.4. In this section, we will construct Green's function $G(x,t,t')$, $t' > 0$, for problem (1.33). $G(x,t,t')$ is the weak solution of (1.33) with the boundary source $f(t) = \delta(t-t')$, where $\delta(t)$ is the Dirac delta-function,

$$\begin{cases} \partial_t^2 G - \partial_x^2 G + qG = 0, \quad 0 < x < l, \quad t > 0, \\ G|_{x=0} = \delta(t-t'), \quad G|_{x=l} = 0, \\ G|_{t=0} = \partial_t G|_{t=0} = 0. \end{cases} \tag{1.37}$$

Lemma 1.12 *Let $G(x,t,t') = G(x,t-t')$ be*

$$G(x,t,t') = \delta(t-t'-x) -$$

$$\frac{1}{2}\int_0^x q(x')dx' \cdot H(t-t'-x) + G_0(x,t-t'). \tag{1.38}$$

Here $H(t)$ is the Heaviside function and $G_0(x,t)$ a continuous function satisfying the initial-boundary value problem,

$$\begin{cases} \partial_t^2 G_0 - \partial_x^2 G_0 + qG_0 = (-\frac{1}{2}q'(x) + \frac{q(x)}{2}\int_0^x q(x')dx')H(t-x) \\ G_0|_{x=0} = G_0|_{x=l} = 0 \\ G_0|_{t=0} = \partial_t G_0|_{t=0} = 0. \end{cases} \tag{1.39}$$

Then $G(x,t,t')$ is the unique solution of initial-boundary value problem (1.37) for $0 \le t \le l$.

Proof. *Step 1.* By Lemma 1.11, initial-boundary value problem (1.39) has a unique solution $G_0(x,t) \in C([0,l]\times[0,l])$. Moreover, $G_0(x,t-t')$ is the unique solution of problem (1.39) with $H(t-x)$ replaced by $H(t-t'-x)$. It is clear that $G(x,t,t')$ of form (1.38) satisfies the initial and boundary conditions required in (1.37). The fact that G satisfies the wave equation (1.37) can be verified by direct substitution.

Step 2. Let $G(x,t,t')$ be the solution of problem (1.37). Then,

$$\widetilde{G}_0(x,t,t') = G(x,t,t') - \delta(t-t'-x) + \frac{1}{2}\int_0^x q(x')dx' \cdot H(t-t'-x),$$

satisfies system (1.39) with $H(t-x)$ replaced by $H(t-t'-x)$. Therefore, by Lemma 1.9,

$$\widetilde{G}_0(x,t,t') = G_0(x,t-t'),$$

which proves the uniqueness. □

In view of Corollary 1.10, $G_0(x,t) = 0$ for $x > t$. Therefore, representation (1.38) implies that $G(x,t,t') = 0$ for $x > t - t'$. We use Green's function only for $0 \le t \le l$, when there is yet no reflection from the right end $x = l$.

Green's function $G(x,t)$ can be used to represent the solution of initial-boundary value problem (1.33) with an arbitrary $f \in L^2([0,T])$,

$$u^f(x,t) = \int_0^t G(x,t,t')f(t')dt' = \tag{1.40}$$

$$= f(t-x) + \int_0^t G_1(x,t')f(t-t')dt',$$

which is valid for $t < l$. Here

$$G_1(x,t) = -\frac{1}{2}\int_0^x q(x')dx' \cdot H(t-x) + G_0(x,t). \quad (1.41)$$

The function G_1 is continuous with respect to (x,t) in the domain $x < t$ and $G_1(x,t) = 0$ for $x > t$.

When $f \in C_0^\infty([0,T])$, we can differentiate representation (1.40) with respect to t. Then,

$$\partial_t^n u^f(x,t) = f^{(n)}(t-x) + \int_0^t G_1(x,t')f^{(n)}(t-t')dt', \quad (1.42)$$

so that $u^f(x,t) \in C^\infty([0,T], L^2([0,l]))$.

1.2.5. In this section, we will obtain a spectral representation of the solution $u^f(x,t)$ of initial-boundary value problem (1.33).

The orthonormalized eigenfunctions $\varphi_k(x)$, $k = 1,2,\ldots,$ of the Schrödinger operator $\mathcal{A}_0$ form an orthonormal basis in $L^2([0,l])$. Since the solution $u^f(x,t)$, $f \in L^2([0,T])$, lies in $L^2([0,l])$ for any t, we can represent it in the form

$$u^f(x,t) = \sum_{k=1}^{\infty} u_k^f(t)\varphi_k(x). \quad (1.43)$$

Here

$$u_k^f(t) = \int_0^l u^f(x,t)\varphi_k(x)dx \quad (1.44)$$

are the Fourier coefficients of $u^f(\cdot,t)$. A simple but very important result is a representation of $u_k^f(t)$ in terms of the boundary spectral data of $\mathcal{A}_0$.

Lemma 1.13 *Let $u^f(x,t)$ be the solution of initial-boundary value problem (1.33) with $f \in L^2([0,T])$. Then, for $k = 1,2,\ldots,$*

$$u_k^f(t) = \int_0^t f(t')s_k(t-t')dt'\varphi_k'(0), \quad (1.45)$$

where

$$s_k(t) = \begin{cases} \frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}, & \lambda_k > 0, \\ t, & \lambda_k = 0, \\ \frac{\sinh\sqrt{|\lambda_k|}t}{\sqrt{|\lambda_k|}}, & \lambda_k < 0. \end{cases} \quad (1.46)$$

Proof. We will give the proof only for $f \in C_0^\infty([0,T])$ and leave the generalization to $f \in L^2([0,T])$ as an exercise. Using (1.44) and integration by parts, we see that

$$\frac{d^2}{dt^2}u_k^f(t) = \int_0^l \partial_t^2 u^f(x,t)\varphi_k(x)dx =$$

$$= \int_0^l \{\partial_x^2 u^f(x,t) - q(x)u^f(x,t)\}\varphi_k(x)dx =$$

$$= [\partial_x u^f(l,t)\varphi_k(l) - u^f(l,t)\partial_x\varphi_k(l)] -$$

$$-[\partial_x u^f(0,t)\varphi_k(0) - u^f(0,t)\partial_x\varphi_k(0)] +$$

$$+ \int_0^l u^f(x,t)\{\partial_x^2\varphi_k(x) - q(x)\varphi_k(x)\}dx, \tag{1.47}$$

where the second equality follows from wave equation (1.22). The boundary conditions for $u^f(x,t)$ and $\varphi_k(x)$ and the fact that φ_k is the eigenfunction of $\mathcal{A}_0$ that corresponds to the eigenvalue λ_k, mean that equation (1.47) takes the form

$$\frac{d^2}{dt^2}u_k^f(t) = -\lambda_k u_k^f(t) + \varphi_k'(0)f(t). \tag{1.48}$$

The initial conditions in (1.33) yield that

$$u_k^f(0) = \frac{d}{dt}u_k^f(0) = 0. \tag{1.49}$$

Solving equation (1.48) with initial conditions (1.49), we obtain formula (1.45) for the Fourier coefficients $u_k^f(t)$. □

Exercise 1.14 *By Lemma 1.11, the map $f \mapsto u^f(\cdot,t)$ is a continuous map from $L^2([0,T])$ into $L^2([0,l])$ for any $0 < t < T$. Use this fact to show the validity of (1.45) for $f \in L^2([0,T])$.*

1.2.6. Lemma 1.13 makes it possible to calculate the inner products of any two waves from the boundary, i.e., solutions of initial-boundary value problem (1.33). Indeed, let $u^f(x,t)$ and $u^h(x,t)$ be the solutions of (1.33) that correspond to the boundary sources f and h. Then, for any $t, s \geq 0$,

$$\langle u^f(\cdot,t), u^h(\cdot,s)\rangle = \sum_{k=1}^{\infty} u_k^f(t)\overline{u_k^h(s)}, \tag{1.50}$$

where the Fourier coefficients in the right-hand side of (1.50) are given by formula (1.45).

1.3. Controllability and projectors

1.3.1. Representation (1.43), (1.45) makes it possible to find the wave $u^f(\cdot,T)$ for arbitrary f. Let us change our point of view and try to answer the following question.

Given a function $a \in L^2([0,l])$, we want to find a source $f \in L^2([0,T])$ such that $u^f(\cdot,T) = a$. Problems of this type are often called problems of controllability. They play a crucial role in the considerations of this book.

First, we note that, due to the finite velocity of the wave propagation, which is equal to 1 in our case,

$$\text{supp } u^f(\cdot,T) \subset [0,T]. \tag{1.51}$$

Hence, for the positive answer to the controllability problem, it is necessary that supp $a \subset [0,T]$.

When $q = 0$, representation (1.40), where $G_1(x,t) = 0$ for $t < l$, shows that $u^f(x,T) = f(T-x)$, $T < l$. This gives an immediate answer to the controllability problem for an arbitrary $a \in L^2([0,T])$, i.e., for $a \in L^2([0,l])$, supp $a \subset [0,T]$, the solution is given by

$$f(t) = a(T-t).$$

This result can be extended to the general case.

Lemma 1.15 *For any $0 \leq T \leq l$ and any $a \in L^2([0,T])$ there exists a unique boundary source $f \in L^2([0,T])$ such that*

$$u^f(x,T) = a(x). \tag{1.52}$$

Proof. Using representation (1.40), we rewrite equation (1.52) in the form

$$a(x) = f(T-x) + \int_0^T G_1(x,t')f(T-t')dt', \quad 0 \le x \le T. \quad (1.53)$$

Changing variables to $y = T - x$, equation (1.53) takes the form

$$a(T-y) = f(y) + \int_0^T G_1(T-y,t')f(T-t')dt'$$

Denote $G_1(T-y, T-t')$ by $G_T(y,t')$. Then, due to the note after the proof of Lemma 1.12, $G_T(y,t') = 0$ for $t' > y$. Moreover, by Lemma 1.12, $G_T(y,t')$ is continuous as a function of (y,t') for $y > t'$. Now we can rewrite equation (1.53) as a Volterra equation of the second kind,

$$f(y) + \int_0^y G_T(y,t')f(t')dt' = a(T-y).$$

This equation has a unique solution $f(y)$, $0 \le y \le T$. It can be obtained by iterations,

$$f_n(y) = a(T-y) - \int_0^y G_T(y,t')f_{n-1}(t')dt',$$

where $f_0(y) = 0$. □

For $T > l$, the solution of controllability problem (1.52) is no longer unique. However, we can still prove the existence.

Corollary 1.16 *For any $T > l$ and any $a \in L^2([0,l])$, there exists a boundary source $f \in L^2([0,T])$, such that*

$$u^f(x,T) = a(x). \quad (1.54)$$

Proof. By Lemma 1.15, for any $a \in L^2([0,l])$ there exists a unique $f \in L^2([0,l])$, such that

$$u^f(x,l) = a(x).$$

Introduce the function

$$\widetilde{f}(t) = \begin{cases} 0, & \text{for} \quad 0 < t < T-l, \\ f(t-(T-l)), & \text{for} \quad T-l < t < T, \end{cases}$$

$\widetilde{f} \in L^2([0,T])$. Since q is time-independent,

$$u^{\widetilde{f}}(x,t) = u^f(x, t-(T-l)),$$

for any $t > 0$. In particular,

$$u^{\widetilde{f}}(x,T) = u^f(x,l) = a(x).$$

□

1.3.2. In this section, we will introduce one of the main tools that we use throughout this book to solve the inverse boundary spectral problem. This is the so-called slicing procedure.

To define a slice, we first fix a point x_0 on the boundary which, in the one-dimensional case, may be either $x = 0$ or $x = l$. Then we take two arbitrary positive number $\tau_2 > \tau_1 \geq 0$. The corresponding slice, $\mathcal{M}(x_0; \tau_1, \tau_2)$, is the set of all points in the domain, for which the travel time to x_0 lies between τ_1 and τ_2.

Since the coordinate x in wave equation (1.22) is the travel time coordinate,

$$\mathcal{M}(0;\tau_1,\tau_2) = [\tau_1,\tau_2] \cap [0,l], \tag{1.55}$$

$$\mathcal{M}(l;\tau_1,\tau_2) = [l-\tau_2, l-\tau_1] \cap [0,l].$$

For general wave equation (1.19),

$$\mathcal{M}(0;\tau_1,\tau_2) = \{x : \tau_1 \leq \int_0^x g^{1/2}(x')dx' \leq \tau_2\} \cap [0,l],$$

$$\mathcal{M}(l;\tau_1,\tau_2) = \{x : l-\tau_2 \leq \int_x^l g^{1/2}(x')dx' \leq l-\tau_1\} \cap [0,l]. \tag{1.56}$$

Since we deal with Problem 2 only, we use notation $\mathcal{M}(\tau_1,\tau_2) = \mathcal{M}(0,\tau_1,\tau_2)$.

Any slice $\mathcal{M}(\tau_1,\tau_2)$ corresponds to a subspace $L^2(\mathcal{M}(\tau_1,\tau_2)) \subset L^2([0,l])$. It consists of all functions with support in $\mathcal{M}(\tau_1,\tau_2)$. The orthoprojector, $P_{\tau_1,\tau_2} = P_{\mathcal{M}(\tau_1,\tau_2)}$ in $L^2([0,l])$ onto $L^2(\mathcal{M}(\tau_1,\tau_2))$ has the form

$$(P_{\tau_1,\tau_2}a)(x) = \chi_{\tau_1,\tau_2}(x)a(x), \tag{1.57}$$

where $\chi_{\tau_1,\tau_2} = \chi_{\mathcal{M}(\tau_1,\tau_2)}(x)$ is the characteristic function of the slice $\mathcal{M}(\tau_1, \tau_2)$,

$$\chi_{\tau_1,\tau_2}(x) = \begin{cases} 1, & \text{for } x \in \mathcal{M}(\tau_1, \tau_2), \\ 0, & \text{for } x \notin \mathcal{M}(\tau_1, \tau_2). \end{cases}$$

Then the orthoprojector P_{τ_1,τ_2} determines the Gram-Schmidt matrix M_{τ_1,τ_2},

$$(M_{\tau_1,\tau_2})_{jk} = \langle P_{\tau_1,\tau_2}\varphi_j, \varphi_k \rangle, \quad j, k = 1, 2, \dots . \tag{1.58}$$

We will show shortly that, for any slice $\mathcal{M}(\tau_1, \tau_2)$, the Gram-Schmidt matrix (1.58) can be constructed from the boundary spectral data. The procedure of constructing this matrix from the boundary spectral data is called the slicing procedure.

1.3.3. We precede the construction of the Gram-Schmidt matrices M_{τ_1,τ_2} with the demonstration of their importance for the solution of the inverse boundary spectral problem. In fact, in the one-dimensional case, the knowledge of $\langle P_{\tau_1,\tau_2}\varphi_j, \varphi_j\rangle$ as a function of (τ_1, τ_2) for any j is sufficient for the reconstruction of the potential q.

Lemma 1.17 *Assume that for some j and any $\tau > 0$, we know $\langle P_{0,\tau}\varphi_j, \varphi_j\rangle$, where φ_j is a normalized eigenfunction of the Schrödinger operator $\mathcal{A}_0$. Then these data uniquely determine l and $q(x)$, $0 < x < l$.*

Proof. By the definition of $P_{0,\tau}$,

$$\langle P_{0,\tau}\varphi_j, \varphi_j\rangle = \int_0^{\min(\tau,l)} \varphi_j^2(x)dx.$$

By this formula, $\langle P_{0,\tau}\varphi_j, \varphi_j\rangle < 1$ for $\tau < l$ and $\langle P_{0,\tau}\varphi_j, \varphi_j\rangle = 1$ for $\tau \geq l$. This observation makes it possible to find l. For $\tau < l$ we use the formula

$$\frac{d}{d\tau}\langle P_{0,\tau}\varphi_j, \varphi_j\rangle = \varphi_j^2(\tau). \tag{1.59}$$

As $\varphi_j(x)$ has no multiple zeros for $0 < x < l$, equation (1.59) determines function $\varphi_j(x)$, $0 \leq x \leq l$ to within a multiplication by ± 1. Then, outside a finite number of points,

$$q(x) = \frac{\partial_x^2\varphi_j(x) + \lambda_j\varphi_j(x)}{\varphi_j(x)}.$$

Since $q(x)$ is smooth, this determines q on the whole interval $[0, l]$.
□

1.3.4. In this section, we will construct the Gram-Schmidt matrix M_{τ_1,τ_2} from the boundary spectral data.

Theorem 1.18 *Let* $\{\lambda_j, \varphi_j'(0) : \ j = 1, 2, \dots\}$ *be the boundary spectral data of a Schrödinger operator* $\mathcal{A}_0$*. Then these data determine the Gram-Schmidt matrix* M_{τ_1,τ_2} *for any* $0 < \tau_1 < \tau_2 < l$.

Proof. The proof will be divided into several steps.

Step 1. By the definition of M_{τ_1,τ_2},

$$M_{\tau_1,\tau_2} = M_{0,\tau_2} - M_{0,\tau_1}.$$

Therefore, it is sufficient to construct $M_{0,\tau}$ for arbitrary $\tau > 0$.

Step 2. Let us take any orthonormal basis $\{\alpha_k(t); k = 1, 2, \dots\}$ in $L^2([0, \tau])$. Denote by $u^{\alpha_k}(x, t)$ the solutions of initial-boundary value problem (1.33) with $f(t) = \alpha_k(t)$, $k = 1, 2, \dots$.

In view of Lemma 1.15, the finite linear combinations of functions $u^{\alpha_k}(\cdot, \tau)$, $k = 1, 2, \dots$, form a dense set in $L^2([0, \min(\tau, l)])$. Moreover, by means of the construction in section 1.2.6, we can find the inner products $\langle u^{\alpha_k}(\cdot, \tau), u^{\alpha_j}(\cdot, \tau)\rangle$, $j, k = 1, 2, \dots$. In general,

$$U_{kj} = \langle u^{\alpha_k}(\cdot, \tau), u^{\alpha_j}(\cdot, \tau)\rangle \neq \delta_{kj}, \tag{1.60}$$

so that $u^{\alpha_k}(\cdot, \tau)$ are not orthonormal.

Step 3. Using the Gram-Schmidt orthogonalization procedure for the matrix $\{U_{kj}\}_{k,j=1}^{\infty}$, we can find finite linear combinations of $u^{\alpha_k}(x, \tau)$,

$$v_l(x) = \sum_{k=1}^{p(l)} d_{kl} u^{\alpha_k}(x, \tau),$$

which form an orthonormal basis in $L^2([0, \min(\tau, l)])$. Since initial-boundary value problem (1.33) is linear,

$$v_l(x) = u^{\beta_l}(x, \tau), \tag{1.61}$$

where

$$\beta_l(t) = \sum_{k=1}^{p(l)} d_{kl} \alpha_k(t).$$

Step 4. Since $\{v_l(x), l = 1, 2, \dots\}$ form an orthonormal basis in $L^2([0, \min(\tau, l)])$, then, for any $a \in L^2([0, l])$,

$$P_{0,\tau} a = \sum_{l=1}^{\infty} \langle a, v_l \rangle v_l(x). \tag{1.62}$$

In particular, this formula implies that

$$\langle P_{0,\tau} \varphi_j, \varphi_k \rangle = \sum_{l=1}^{\infty} \langle \varphi_j, v_l \rangle \overline{\langle \varphi_k, v_l \rangle}. \tag{1.63}$$

Furthermore, the inner products $\langle v_l, \varphi_j \rangle$ are the Fourier coefficients of $v_l(x)$ with respect to the basis $\{\varphi_j(x), j = 1, 2, \dots\}$. By Lemma 1.13,

$$\langle v_l, \varphi_j \rangle = \int_0^{\tau} \beta_l(t') s_j(\tau - t') dt' \varphi_j'(0),$$

where $s_j(t)$ are given by formulae (1.46). □

Hence Theorem 1.6 is proven.

1.3.5. To conclude section 1.3, we will show that the Gram-Schmidt matrix M_{τ_1,τ_2} determines the inner products $\langle P_{\tau_1,\tau_2} u^f(\cdot, t), u^g(\cdot, s) \rangle$ for any sources f, g and $t, s > 0$. This result is used later to provide another solution of the one-dimensional inverse boundary spectral problem.

Lemma 1.19 *For any sources $f, g \in L^2([0, \infty))$,*

$$\langle P_{\tau_1,\tau_2} u^f(\cdot, t), u^g(\cdot, s) \rangle = \sum_{j,k=1}^{\infty} (M_{\tau_1,\tau_2})_{jk} u_j^f(t) \overline{u_k^g(s)}, \tag{1.64}$$

where $u_j^k(t)$ and $u_k^g(s)$ are given by formulae (1.45).

Exercise 1.20 *Prove formula (1.64).*

1.4. Gaussian beams

1.4.1. In this section, we will introduce the second main tool, which is also used later to solve the multidimensional inverse boundary spectral problem. As it was shown in section 1.3, the slicing procedure is sufficient to reconstruct the potential q. To illustrate the difference between the one-dimensional and the multidimensional cases, we mention two things. Both of them are of geometric character.

i) In the one-dimensional case, the slice $\mathcal{M}(\tau_1, \tau_2)$ is known a priori for each τ_1 and τ_2. This follows from the fact that, in the one-dimensional case, any operator can be transformed to a Schrödinger operator. Then, in the travel-time coordinates, the corresponding slice is just the interval $\tau_1 \leq x \leq \tau_2$. In the multidimensional case, it is, in principle, impossible to transform a general second-order operator to the form

$$-\left((\frac{\partial}{\partial x^1})^2 + \cdots + (\frac{\partial}{\partial x^m})^2\right) + q,$$

so that the boundary spectral data remain intact. Henceforth, the geometry of slices are *a priori* unknown.

ii) In the one-dimensional case, the slices $\mathcal{M}(\tau_1, \tau_2)$ shrink to x_0, if $\tau_1 < x_0 < \tau_2$ and $\tau_1, \tau_2 \to x_0$. This makes it possible to isolate the point x_0. We have used this localization to find $q(x)$. In the multidimensional case, when $\tau_1, \tau_2 \to \tau_0$, the corresponding slices $\mathcal{M}(\tau_1, \tau_2)$ do not, in general, tend to one point. There is a way to develop an analogous slicing procedure to get localization in the multidimensional case. However, we prefer to overcome this difficulty by using some special solutions of the wave equation that have the localization property. In this section, we will construct some special solutions of the wave equation which, at any time t, are localized near a point $x = x(t)$. These solutions are called Gaussian beams or quasiphotons. The point of localization, $x = x(t)$, moves along a characteristic of the wave equation.

1.4.2. Consider initial-boundary value problem (1.33), where the source f is of the form

$$f(t) = f_\epsilon(t; a) = (\pi\epsilon)^{-1/4} \exp\{-(i\epsilon)^{-1}\theta(t-a)\}\chi_a(t). \qquad (1.65)$$

Here

$$\theta(t) = -t + \frac{i}{2}t^2, \qquad (1.66)$$

is called the phase function and a and ϵ are positive constants, and the function $\chi_a(t)$ is given by the formula,

$$\chi_a(t) = \chi\left(\frac{t-a}{a}\right),$$

where $\chi(t)$ is a usual smooth cut-off function,

$$\chi(t) = \begin{cases} 1, & \text{for } |t| < 1/2, \\ 0, & \text{for } |t| > 1 \end{cases}, \quad 0 \le \chi(x) \le 1, \quad \chi \in C^\infty(\mathbf{R}).$$

Definition 1.21 *The Gaussian beam $u_\epsilon(x,t;a)$ is the solution of initial-boundary value problem (1.33) with the source f_ϵ of the form (1.65), (1.66).*

Usually, Gaussian beams are defined as a special class of asymptotic solutions of the wave equation. We use only Gaussian beams of a special type. Their asymptotic properties are described in the following theorem.

Theorem 1.22 *Let $u_\epsilon(x,t;a)$ be a Gaussian beam. Then for $t < l+a$*

$$u_\epsilon(x,t;a) = \tag{1.67}$$

$$(\pi\epsilon)^{-1/4}\chi_a(t-x)\exp\{-(i\epsilon)^{-1}\theta(t-x-a)\}(1+i\epsilon u_1(x,t-a))+$$

$$+R_\epsilon(x,t;a),$$

where

$$u_1(x,t) = \frac{1}{2\theta'(t-x)}\int_0^x q(x')dx'. \tag{1.68}$$

The remainder $R_\epsilon(x,t;a)$ satisfy the estimate

$$\|R_\epsilon(\cdot,t;a)\| \le C\epsilon^2, \tag{1.69}$$

where the constant C may depend on a and $t < l+a$.

Proof. We construct the Gaussian beam by using the anzatz,

$$U_\epsilon^N(x,t) = (\pi\epsilon)^{-1/4} \exp\{-(i\epsilon)^{-1}\theta(t-x)\} \sum_{n=0}^{N} u_n(x,t)(i\epsilon)^n. \quad (1.70)$$

Following the ideology of asymptotic methods, we consider ϵ as an independent small parameter.

Step 1. When we substitute $U_\epsilon^N(x,t)$ into wave equation (1.22) and obtain a polynomial of ϵ with coefficients depending on (x,t),

$$\partial_t^2 U_\epsilon^N - \partial_x^2 U_\epsilon^N + q(x) U_\epsilon^N =$$

$$= \pi^{-1/4}\epsilon^{-5/4} \exp\{-(i\epsilon)^{-1}\theta(t-x)\} \sum_{n=0}^{N+1} v_n(x,t)(i\epsilon)^n, \quad (1.71)$$

where

$$v_n = i\{2\theta'(t-x)(\partial_x + \partial_t)u_n(x,t) - (\partial_t^2 - \partial_x^2 + q(x))u_{n-1}(x,t)\},$$

for $n = 0, 1, \ldots, N+1$, and $u_{-1} \equiv 0$, $u_{N+1} \equiv 0$.

Next we will find the $N+1$ unknown functions $u_0, \ldots, u_N$, using the $N+1$ equations $v_0 = 0, \ldots, v_N = 0$. Note that we do not pose any conditions for v_{N+1}. This will minimize the right-hand side of equation (1.71) when $\epsilon \to 0$. The requirement that $v_n(x,t) = 0$ for $n = 0, 1, \ldots, N$, yields a recurrent system of equations for $u_n(x,t)$,

$$\frac{\partial u_n}{\partial t}(y,t) = \frac{-1}{2\theta'(-y)} \left\{ 2\frac{\partial^2 u_{n-1}}{\partial y \partial t} - \frac{\partial^2 u_{n-1}}{\partial t^2} - q(y+t)u_{n-1} \right\} \quad (1.72)$$

where we use the coordinates $y = x - t$ and t.

To satisfy boundary condition (1.65), we require that

$$u_0(y,-y) = 1, \quad u_n(y,-y) = 0, \quad \text{for } n = 1, \ldots, N. \quad (1.73)$$

which in coordinates (x,t) correspond to the conditions

$$u_0(0,t) = 1, \quad u_n(0,t) = 0, \quad \text{for} \quad n = 1, \ldots, N.$$

Equations (1.72) are ordinary differential equations in t for the functions $u_n(y,t)$, where y is considered as a parameter. Therefore, system (1.72), (1.73) constitutes a Cauchy problem for u_n. Since

$u_{-1} \equiv 0$, we can solve these equations recurrently, starting from $n = 0$. In particular, we obtain that $u_0 \equiv 1$ and u_1 have form (1.68).

Next we use U_ϵ^N to construct the Gaussian beam $u_\epsilon(x, t; a)$ and prove estimate (1.69).

Step 2. Consider the function $\chi_a(t - x)U_\epsilon^2(x, t - a)$ that corresponds to $N = 2$. This function is the solution of the initial-boundary value problem,

$$\left(\partial_t^2 - \partial_x^2 + q\right)\left[\chi_a(t - x)U_\epsilon^2(x, t - a)\right] = \mathcal{F}_{\epsilon,a}(x, t), \qquad (1.74)$$

$$\chi_a(t - x)U_\epsilon^2(x, t - a)\big|_{x=0} = f_\epsilon(t, a), \;\; \chi_a(t - x)U_\epsilon^2(x, t - a)\big|_{x=l} = 0,$$

$$\chi_a(t - x)U_\epsilon^2(x, t - a)\big|_{t=0} = \partial_t[\chi_a(t - x)U_\epsilon^2(x, t - a)]\big|_{t=0} = 0,$$

for any $t \le l$. Here $\mathcal{F}_{\epsilon,a}(x, t)$ is a function that satisfies the inequality,

$$\|\mathcal{F}_{\epsilon,a}\|_{L^2([0,l]\times[0,l])} \le C_a \epsilon^2. \qquad (1.75)$$

Estimate (1.75) follows from two observations.

First, for any $M > 0$,

$$|\exp\{-(i\epsilon)^{-1}\theta(y)\}(1 - \chi(y))| \le C\epsilon^M$$

and the same is true for all derivatives of this function. Therefore, multiplication of $U_\epsilon^2(x, t)$ by $\chi_a(t - x)$ gives an error of order ϵ^M for any $M > 0$ to (1.33).

Second, due to the constructions of step 1, U_ϵ^2 satisfies the equation

$$\left(\partial_t^2 - \partial_x^2 + q\right)U_\epsilon^2 = \pi^{-\frac{1}{4}}\epsilon^{-\frac{5}{4}}\exp\{-(i\epsilon)^{-1}\theta(t - x - a)\}v^3(x, t)(i\epsilon)^3.$$

Step 3. Comparing initial-boundary value problem (1.33) with $f = f_\epsilon(t, a)$ of form (1.65) and initial-boundary value problem (1.74), it follows from Lemma 1.11 that,

$$\|u_\epsilon(\cdot, t; a) - \chi_a(t - \cdot)U_\epsilon^2(\cdot, t - a)\|_{L^2([0,l])} \le C_a\epsilon^2,$$

for any $t \le l$.

At last, we observe that in the sum (1.70) with $N = 2$, the last term satisfies

$$\|\epsilon^{-\frac{1}{4}}\chi_a(t - x)\exp\{-(i\epsilon)^{-1}\theta(x - t + a)\}(i\epsilon)^2 u_2(x, t)\|_{L^2([0,l])} \le C_a\epsilon^2$$

for any $t \le l$.

Summarizing, we obtain representation (1.67), (1.68) with the remainder estimate (1.69). □

Using the representation of the Gaussian beam given by Theorem 1.22, we see that

$$\|u_\epsilon(\cdot, t; a)\| = 1 + O(\epsilon^2), \quad \text{for } a < t < l. \tag{1.76}$$

Exercise 1.23 *Prove estimate (1.76).*

1.4.3. In this and the next sections we will combine the slicing procedure and the technique of Gaussian beams to find $q(x)$.

Lemma 1.24 *Let $u_\epsilon(x, t; a)$ be a Gaussian beam. Then, for any t, τ, a, ϵ, such that*

$$0 < a < t < l, \quad \tau < t - a, \quad 0 < \epsilon < 1,$$

we have

$$\langle P_{0,\tau} u_\epsilon(\cdot, t; a), u_\epsilon(\cdot, t; a)\rangle = O(\epsilon^2), \quad \tau < t - a,$$

$$\langle P_{0,\tau} u_\epsilon(\cdot, t; a), u_\epsilon(\cdot, t; a)\rangle = 1 + O(\epsilon^2), \quad \tau > t - a, \tag{1.77}$$

$$\langle P_{0,\tau} u_\epsilon(\cdot, t; a), u_\epsilon(\cdot, t; a)\rangle = \frac{1}{2} + \frac{\epsilon^{3/2}}{2\sqrt{\pi}} \int_0^\tau q(x)dx + O(\epsilon^2), \quad \tau = t - a.$$

Note that the left-hand side is continuous with respect to t, τ, a, ϵ. On the other hand, the right-hand side is discontinuous with respect to t, τ, a. This means that asymptotic formula (1.77) is not uniform with respect to t, τ, a. It actually describes the behaviour of the inner product as a function of ϵ, $\epsilon \to 0$, for fixed parameters t, τ, a.

Proof. The first two inequalities of formula (1.77) for $\tau < t - a$ and $\tau > t - a$, can be obtained by the following considerations.

First, in view of the exponential factor $\exp\{-(i\epsilon)^{-1}\theta(t - x - a)\}$, the Gaussian beam $u_\epsilon(x, t; a)$ decays exponentially when $\epsilon \to 0$ and $x \ne t - a$. Second, by formula (1.76), the L^2-norm of the Gaussian beam is $1 + O(\epsilon^2)$.

To prove formula (1.77) for $\tau = t - a$, we use the asymptotic expansion given by Theorem 1.22. Due to estimate (1.69) of $R_\epsilon(x,t;a)$ and the exponential decay for the Gaussian beam,

$$\langle P_{0,t-a} u_\epsilon(\cdot,t;a), u_\epsilon(\cdot,t;a)\rangle =$$

$$= \int_{-\infty}^{t-a} |U_\epsilon^1(x,t-a)|^2 dx + O(\epsilon^2). \tag{1.78}$$

Here U_ϵ^1 is given by formula (1.70) with $N = 1$. By means of representation (1.67), (1.68), the integral in the right-hand side of formula (1.78) can be written in the form

$$(\pi\epsilon)^{-1/2} \int_{-\infty}^{0} e^{-y^2/\epsilon}(1 - 2\epsilon \Im u_1(y+t-a, t-a))dy + O(\epsilon^2) =$$

$$=(\pi\epsilon)^{-\frac{1}{2}} \int_{-\infty}^{0} e^{-\frac{y^2}{\epsilon}}(1 - \epsilon \Im \left(\frac{1}{\theta'(-y)} \int_0^{y+t-a} q(x)dx\right))dy + O(\epsilon^2), \tag{1.79}$$

where $y = x - t + a$. Using representation (1.66) for the phase function θ, we see that

$$\Im \left(\frac{1}{\theta'(-y)} \int_0^{y+t-a} q(x)dx\right) = y \int_0^{t-a} q(x)dx + O(y^2). \tag{1.80}$$

Combining estimates (1.79), (1.80), we obtain that

$$(\pi\epsilon)^{-1/2} \int_{-\infty}^{0} e^{-y^2/\epsilon} \left[1 - \epsilon \Im \left(\frac{1}{\theta'(-y)} \int_0^{y+t-a} q(x)dx\right)\right] dy =$$

$$= \frac{1}{2} + \frac{\epsilon^{3/2}}{2\sqrt{\pi}} \int_0^{t-a} q(x)dx + O(\epsilon^2).$$

Summarizing the previous calculations, we prove formula (1.77) for $\tau = t - a$.

□

1.4.4. We are now in the position to describe a procedure to solve the inverse boundary spectral problem.

i) By means of the slicing procedure described in section 1.3.4, we find the Gram-Schmidt matrix $M_{0,\tau}$ for any $\tau > 0$.

ii) By means of the procedure described in section 1.3.5, we calculate the inner products

$$\langle P_{0,\tau} u_\epsilon(\cdot, \tau + a; a), u_\epsilon(\cdot, \tau + a; a)\rangle.$$

iii) Using these inner products, we obtain from Lemma 1.24 that

$$\int_0^t q(x)dx = \lim_{\epsilon \to 0} \frac{2\sqrt{\pi}}{\epsilon^{3/2}} \left\{ \langle P_{0,t} u_\epsilon(\cdot, t + a; a), u_\epsilon(\cdot, t + a; a)\rangle - \frac{1}{2} \right\} \tag{1.81}$$

for $0 < t < l - a$.

iv) Differentiating integral (1.81), we find $q(x)$ for $0 \leq x \leq l - a$. Since $a > 0$ is arbitrary, $q(x)$ is found for $0 \leq x \leq l$.

Notes. There are numerous textbooks on the spectral theory of the Sturm-Liouville operators. The properties of the eigenvalues and eigenfunctions of these operators, and, in particular, their asymptotic properties, which are used in sections 1.1.2 and 1.1.7, can be found, e.g., in [Ki], section 4.3, [LeSa], Chapter 1, [ReRg], Section 10.2, [ChCoPaRn], Chapter 3. The precise formulation of the initial-boundary value problem for the wave equation and its main properties, such as existence, uniqueness, regularity and finite velocity of the wave propagation, can be found, e.g., in [Ld], Chapter 4, [Ev], Section 7.2. These textbooks deal with the multidimensional case and we use them also as our standard references for the multidimensional wave equation. The theory of the one-dimensional Gaussian beams, which is described in section 1.4, is self-contained. For those readers who are interested in further developments of this theory, we can recommend [BaBuMo], Chapter. 3. The basic theory of Volterra equations used in Chapter 1 can be found, e.g., in [ChCoPaRn].

The one-dimensional inverse boundary spectral problem goes back to the classical works [Am], [Bo], [Lv], [GeLe], [Kr1]–[Kr4], [Ma1]–[Ma4]. There are now numerous efficient methods to solve this problem and also other types of the one-dimensional inverse problems. We refer only to some major classical and modern textbooks on the subject, such as [PoTr], [Ma1], [Ma2], [LeSa], [Nz], [Le], [ChSa], [BeBl1], [Ki]. Our approach, which is developed in this chapter, has its roots in the method of Krein-Blagovestchenskii, [Kr1]–[Kr4] and [Bl1]–[Bl3].

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Table of notation

Notation	Meaning	See page
$\mathbf{R}, \mathbf{C}$	Real and complex numbers	
$\Re, \Im$	Real and imaginary parts	
$\mathcal{N}, \mathcal{M}$	Manifolds, $\partial\mathcal{M} \neq \emptyset$	??
$H^s(\mathcal{M}), H^s_0(\mathcal{M}),$	Sobolev spaces	??, ??
$\\| \cdot \\|_s, \\| \cdot \\|_{(s,\partial M)}$	Sobolev s-norms	??
$\langle \cdot , \cdot \rangle, \ \\| \cdot \\|$	Inner product, norm in L^2	3,??
$C^p, L^p([0,T]; H^s)$	H^s-valued functions	15, ??
$\|\|\|\cdot\|\|\|$	Norm of $L^1([0,T]; L^2(\mathcal{M}))$	??
$C_0^\infty, \mathring{C}^\infty$	Smooth function spaces	??
$T_\mathbf{x}\mathcal{N}, T^*_\mathbf{x}\mathcal{N}, S\mathcal{N}, S^*\mathcal{N}$	Tangent space, etc.	??,??,??
$(\mathbf{v}, \mathbf{v})_g, (\mathbf{p}, \mathbf{q})_g, (\mathbf{p}, \mathbf{v})$	Inner products, duality	??, ??
$(\cdot , \cdot)$	Also, inner product of $\mathbf{R}^m$	??
$\{ \cdot , \cdot \}$	Poisson brackets	??
$B_\rho(\mathbf{y})$	Ball in $(\mathcal{M}, g)$	??
$g_{ik}, \Gamma_{i,kl}, R^i_{jkl}$	Metric, Christoffel, curvature	??,??,??
$\gamma_{\mathbf{z},\mathbf{w}}, \gamma_{\mathbf{z},\nu}$	Geodesics	??
$\exp, \exp_{\partial\mathcal{M}}$	Exponential mappings	??,??
$l(\mathbf{z}_0)$	Length of $\gamma_{\mathbf{z},\nu}$	??
$\tau(\mathbf{x}, \xi), \tau_{\partial M}(\mathbf{z})$	Critical values on geodesics	??,??
$\omega(\mathbf{z}_0), \omega_{\partial\mathcal{M}}$	Cut loci	??,??
$\nabla_\mathbf{v}\mathbf{w}, \frac{D\mathbf{w}}{ds}$	Covariant derivative	??,??
$\mathcal{M}(\tau_1, \tau_2)$	Slices	21
$\mathcal{M}^T$,$\mathcal{M}(\Gamma, \tau)$,$\mathcal{M}(\mathbf{z}, \tau)$	Domains of influence	??,??,??
$D^T(\Omega), K_{\Gamma,T}$	Cone, double cone	??, ??
Q^T, Σ^T	Time-domains	??
$\mathcal{A}, \mathcal{A}_0, \mathcal{A}_\kappa$	Operators	6,7,??
$a(\mathbf{x}, D), p(\mathbf{x}, D)$	Differential expressions	??,??
dV_g, dV_{can}	Volume elements	??,??
$P_{\Gamma,\tau}, P_{\mathbf{z},\tau}, P_{\tau_1,\tau_2}$	Projections onto $\mathcal{M}(\Gamma, \tau)$	21,??
u_ϵ, U^N_ϵ	Gaussian beams	??, ??
$\theta(\mathbf{x}, t), u_n(\mathbf{x}, t)$	Phase and amplitudes	??
f^ϵ, M_ϵ	Boundary source of u_ϵ, etc.	??,??
$R, r_\mathbf{x}, R^T, r^T_\mathbf{x}$	Boundary distance functions	??
$\Pi, \Lambda, \mathcal{B}, L, \Lambda_\lambda$	Boundary maps/forms	??
$\sigma(\mathcal{A}), \sigma_0(\mathcal{A})$	Orbits of gauge group	??,??