Mat-1.3100 Funktionaalianalyysin perusteet Harj. (2)

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1) Tutustu Hölderin epäyhtälön $\sum_{k=1}^{\infty} x_k y_k \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{1/q}$ todistukseen (liitteenä) jonoille $(x_k) \in \ell^p$, $(y_k) \in \ell^q$, $1 < p, q < \infty$ $\frac{1}{p} + \frac{1}{q} = 1$.

2) Osoita, että edellisen tehtävän epäyhtälö on yhtälö jos ja vain jos löytyy vakio C_1 siten, että $|y_k|^q = C_1 |x_k|^p$ kaikilla $k \in \mathbb{N}$.

3) Tutustu Minkowskin epäyhtälön $\left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}$ todistukseen (liitteenä) jonoille $(x_k), (y_k) \in \ell^p$ $1 \le p < \infty$.

4) Osoita, että edellisen tehtävän epäyhtälö on yhtälö jos ja vain jos löytyy vakio $C \ge 0$ siten, että $x_k = C y_k$ kaikilla $k \in \mathbb{N}$.

5) Selitä (kurssin Moda tietojen pohjalta) käsitteet

a) Lebesguen mitallinen joukko

b) yksinkertainen funktio $f = \sum_{k=1}^{m} \alpha_k \chi_{A_k}$

c) yksinkertaisen funktion f Lebesguen integraali $\int f d\lambda = \sum_{k=1}^{n} \alpha_k \lambda(A_k)$

d) Jos $f \ge 0$ mitallinen niin $\int f d\lambda := \lim_{n \to \infty} \int f_n \, d\lambda$ missä $f(x) = \lim_{n \to \infty} f_n(x)$ ja $0 \le f_n \le f_{n+1} \le \dots \le f$

f_n yksinkertaisia funktioita

e) Jos f mitallinen ja $\int |f| d\lambda < \infty$ niin f on integroituva ja $\int f d\lambda := \int f_+ d\lambda - \int f_- d\lambda$

$f_+(x) = \max\{f(x), 0\}$, $f_-(x) = \max\{-f(x), 0\}$

6) Asetetaan $L^{(p)}(\Omega) = \{ f : \Omega \to \mathbb{R} \mid f$ mitallinen, $\|f\|_p := \left(\int |f(x)|^p d\lambda\right)^{1/p} < \infty \}$

($\Omega \subset \mathbb{R}^n$ mitallinen)

Osoita $L^{(p)}$-avaruuden Hölderin epäyhtälö:

$$\int |(fg)(x)| d\lambda \le \left(\int |f(x)|^p d\lambda\right)^{1/p} \left(\int |g(x)|^q d\lambda\right)^{1/q}$$

Then (7) is satisfied, so that we may apply (8). Substituting (9) into (8) and multiplying the resulting inequality by the product of the denominators in (9), we arrive at the **Hölder inequality** *for sums*

(10) $$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leqq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{1/q}$$

where $p > 1$ and $1/p + 1/q = 1$. This inequality was given by O. Hölder (1889).

If $p = 2$, then $q = 2$ and (10) yields the **Cauchy-Schwarz inequality** *for sums*

(11) $$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leqq \sqrt{\sum_{k=1}^{\infty} |\xi_k|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_m|^2}.$$

It is too early to say much about this case $p = q = 2$ in which p equals its conjugate q, but we want to make at least the brief remark that this case will play a particular role in some of our later chapters and lead to a space (a Hilbert space) which is "nicer" than spaces with $p \neq 2$.

(c) We now prove the **Minkowski inequality** *for sums*

(12) $$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p\right)^{1/p} \leqq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} + \left(\sum_{m=1}^{\infty} |\eta_m|^p\right)^{1/p}$$

where $x = (\xi_j) \in l^p$ and $y = (\eta_j) \in l^p$, and $p \geqq 1$. For finite sums this inequality was given by H. Minkowski (1896).

For $p = 1$ the inequality follows readily from the triangle inequality for numbers. Let $p > 1$. To simplify the formulas we shall write $\xi_j + \eta_j = \omega_j$. The triangle inequality for numbers gives

$$\begin{aligned} |\omega_j|^p &= |\xi_j + \eta_j| |\omega_j|^{p-1} \\ &\leqq (|\xi_j| + |\eta_j|) |\omega_j|^{p-1}. \end{aligned}$$

Summing over j from 1 to any fixed n, we obtain

(13) $$\sum |\omega_j|^p \leqq \sum |\xi_j| |\omega_j|^{p-1} + \sum |\eta_j| |\omega_j|^{p-1}.$$

To the first sum on the right we apply the Hölder inequality, finding

$$\sum |\xi_j| |\omega_j|^{p-1} \leqq \left[\sum |\xi_k|^p\right]^{1/p} \left[\sum (|\omega_m|^{p-1})^q\right]^{1/q}.$$

On the right we simply have

$$(p-1)q = p$$

because $pq = p + q$; see (5). Treating the last sum in (13) in a similar fashion, we obtain

$$\sum |\eta_j| |\omega_j|^{p-1} \leqq \left[\sum |\eta_k|^p\right]^{1/p} \left[\sum |\omega_m|^p\right]^{1/q}.$$

Together,

$$\sum |\omega_j|^p \leqq \left\{\left[\sum |\xi_k|^p\right]^{1/p} + \left[\sum |\eta_k|^p\right]^{1/p}\right\} \left(\sum |\omega_m|^p\right)^{1/q}.$$

Dividing by the last factor on the right and noting that $1 - 1/q = 1/p$, we obtain (12) with n instead of ∞. We now let $n \longrightarrow \infty$. On the right this yields two series which converge because $x, y \in l^p$. Hence the series on the left also converges, and (12) is proved.

(d) From (12) it follows that for x and y in l^p the series in (2) converges. (12) also yields the triangle inequality. In fact, taking any $x, y, z \in l^p$, writing $z = (\zeta_j)$ and using the triangle inequality for numbers and then (12), we obtain

$$\begin{aligned} d(x, y) &= \left(\sum |\xi_j - \eta_j|^p\right)^{1/p} \\ &\leqq \left(\sum [|\xi_j - \zeta_j| + |\zeta_j - \eta_j|]^p\right)^{1/p} \\ &\leqq \left(\sum |\xi_j - \zeta_j|^p\right)^{1/p} + \left(\sum |\zeta_j - \eta_j|^p\right)^{1/p} \\ &= d(x, z) + d(z, y). \end{aligned}$$

This completes the proof that l^p is a metric space. ∎

The inequalities (10) to (12) obtained in this proof are of general importance as indispensable tools in various theoretical and practical problems, and we shall apply them a number of times in our further work.

where $y=(\eta_j)$ and $\sum|\eta_j|^p<\infty$. If we take only real sequences [satisfying (1)], we get the *real space* l^p, and if we take complex sequences [satisfying (1)], we get the *complex space* l^p. (Whenever the distinction is essential, we can indicate it by a subscript **R** or **C**, respectively.)

In the case $p=2$ we have the famous *Hilbert sequence space* l^2 with metric defined by

$$d(x, y)=\sqrt{\sum_{j=1}^{\infty}|\xi_j-\eta_j|^2}. \tag{3}$$

This space was introduced and studied by D. Hilbert (1912) in connection with integral equations and is the earliest example of what is now called a *Hilbert space*. (We shall consider Hilbert spaces in great detail, starting in Chap. 3.)

We prove that l^p is a metric space. Clearly, (2) satisfies (M1) to (M3) provided the series on the right converges. We shall prove that it does converge and that (M4) is satisfied. Proceeding stepwise, we shall derive

(*a*) an auxiliary inequality,
(*b*) the Hölder inequality from (*a*),
(*c*) the Minkowski inequality from (*b*),
(*d*) the triangle inequality (M4) from (*c*).

The details are as follows.

(a) Let $p>1$ and define q by

$$\frac{1}{p}+\frac{1}{q}=1. \tag{4}$$

p and q are then called **conjugate exponents.** This is a standard term. From (4) we have

$$1=\frac{p+q}{pq}, \qquad pq=p+q, \qquad (p-1)(q-1)=1. \tag{5}$$

Hence $1/(p-1)=q-1$, so that

$$u=t^{p-1} \qquad \text{implies} \qquad t=u^{q-1}.$$

Let α and β be any positive numbers. Since $\alpha\beta$ is the area of the rectangle in Fig. 5, we thus obtain by integration the inequality

$$\alpha\beta \leqq \int_0^\alpha t^{p-1}\,dt+\int_0^\beta u^{q-1}\,du=\frac{\alpha^p}{p}+\frac{\beta^q}{q}. \tag{6}$$

Note that this inequality is trivially true if $\alpha=0$ or $\beta=0$.

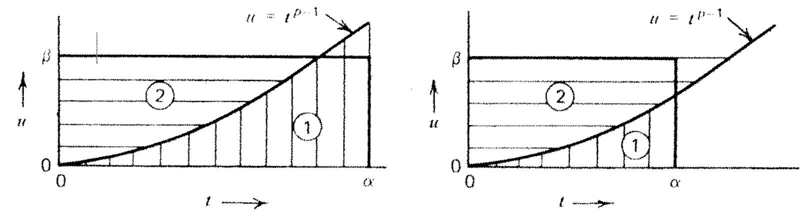


Fig. 5. Inequality (6), where ① corresponds to the first integral in (6) and ② to the second

(b) Let $(\tilde{\xi}_j)$ and $(\tilde{\eta}_j)$ be such that

$$\sum|\tilde{\xi}_j|^p=1, \qquad \sum|\tilde{\eta}_j|^q=1. \tag{7}$$

Setting $\alpha=|\tilde{\xi}_j|$ and $\beta=|\tilde{\eta}_j|$, we have from (6) the inequality

$$|\tilde{\xi}_j\tilde{\eta}_j|\leqq\frac{1}{p}|\tilde{\xi}_j|^p+\frac{1}{q}|\tilde{\eta}_j|^q.$$

If we sum over j and use (7) and (4), we obtain

$$\sum|\tilde{\xi}_j\tilde{\eta}_j|\leqq\frac{1}{p}+\frac{1}{q}=1. \tag{8}$$

We now take any nonzero $x=(\xi_j)\in l^p$ and $y=(\eta_j)\in l^q$ and set

$$\tilde{\xi}_j=\frac{\xi_j}{\left(\sum|\xi_k|^p\right)^{1/p}}, \qquad \tilde{\eta}_j=\frac{\eta_j}{\left(\sum|\eta_m|^q\right)^{1/q}}. \tag{9}$$

2. Hölderin epäyhtälö jonoille: Kaikille $x \in \ell^p$, $y \in \ell^q$,

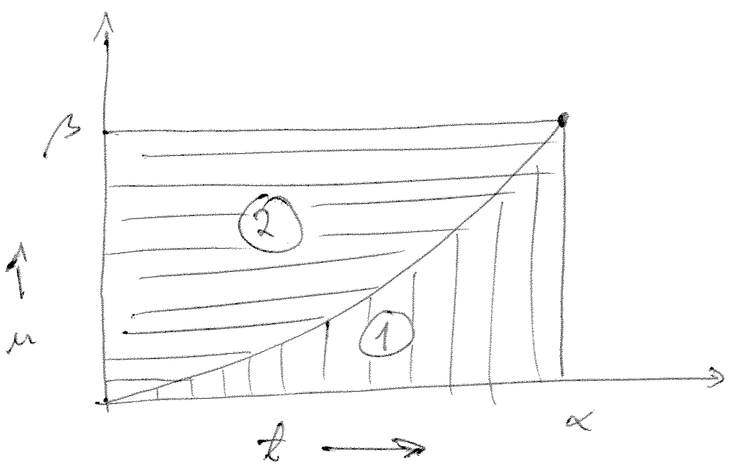
$$\sum_{j=1}^{\infty} |x_j y_j| \leq \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{\infty} |y_j|^q\right)^{1/q},$$

missä $1/p + 1/q = 1$.

Kirjan todistuksessa käytetään mm. yhtälöä

epäyhtälöä: $\alpha\beta \leq \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q$,

missä $\alpha, \beta \geq 0$ ja $\frac{1}{p} + \frac{1}{q} = 1$.



Oheisen kuvan mukaisesti käy jos ja vain jos $\beta = \alpha^{p-1}$.

Koska $(p-1)q = p$, niin saadaan yhtäpitävä ehto $\beta^q = \alpha^p$. Joten

epäyhtälössä (8) elisi yhtälö on oltava

$|\tilde{\eta}_j|^q = |\tilde{\xi}_j|^p$ kaikille $j \in \mathbb{N}$.

Olkoon nyt $x \in \ell^p$ ja $y \in \ell^q$ s.e.

eräälle $C \geq 0$, $C|x_k|^p = |y_k|^q$ $\forall k \in \mathbb{N}$.

Tällöin kirjan merkinnöillä

$$|\tilde{\eta}_j|^q = \left|\frac{\eta_j}{(\sum|\eta_k|^q)^{1/q}}\right|^q$$

$$= \left(\sum|y_k|^q\right)^{-1} |y_j|^q$$

$$= \left(\sum C|x_k|^p\right)^{-1} C|x_j|^p$$

$$= \left|\frac{x_j}{(\sum|x_k|^p)^{1/p}}\right|^p = |\tilde{\xi}_j|^p .$$

Toisaalta, jos $|\tilde{\eta}_j|^q = |\tilde{\xi}_j|^p$ $\forall j \in \mathbb{N}$, niin

$$|y_j|^q = |\eta_j|^q = \|y\|_q^q \, |\tilde{\xi}_j|^p$$

$$= \|y\|_q^q \, \|x\|_p^{-p} \, |\xi_j|^p$$

$$= \|y\|_q^q \, \|x\|_p^{-p} \, |x_j|^p \quad \forall j \in \mathbb{N}.$$

Siis on olemassa $C := \|y\|_q^q \, \|x\|_p^{-p}$ s.e.

$|y_j|^q = C|x_j|^p$ $\forall j \in \mathbb{N}$. <u>Huom!</u> Voidaan olettaa, että

$x, y \neq 0$, sillä muutoin väite on heti tosi.

4. Minkowskin epäyhtälön todistuksessa käytetään kolmioepäyhtälöä ja kahdesti Hölderin epäyhtälöä. Jotta yhtälö toteutuisi on oltava

1) $|\xi_j + \eta_j| = |\xi_j| + |\eta_j| \quad \forall j \in \mathbb{N}$

2) On olemassa $C_1 \geq 0$ s.e.

$$|\xi_j|^p = C_1(|\omega_j|^{p-1})^q$$

3) On olemassa $C_2 \geq 0$ s.e.

$$|\eta_j|^p = C_2(|\omega_j|^{p-1})^q .$$

Tutkitaan milloin 1), 2) ja 3) voimassa.

1) $|\xi_j + \eta_j| = |\xi_j| + |\eta_j| \iff \xi_j = \alpha \eta_j$,

missä $\alpha \geq 0$.

2) $|\xi_j|^p = C_1(|\omega_j|^{p-1})^q \iff |\xi_j|^p = C_1 |\xi_j + \eta_j|^p$

$\overset{1)}{\iff} |\xi_j|^p = C_1(|\xi_j| + |\eta_j|)^p \iff |\xi_j| = C_1^{1/p}(|\xi_j| + |\eta_j|)$

$\iff |\eta_j| = (C_1^{-1/p} - 1)|\xi_j| \iff \exists C > 0 : |\eta_j| = C|\xi_j|$

3) Kuten 2), jolloin saadaan $3) \iff \exists C > 0 : |\xi_j| = C|\eta_j|$.

Siis 1), 2) ja 3) samaan aikaan voimassa tarkoittaa samaa kuin: $\exists C \geq 0$ s.e. $y_k = C x_k \quad \forall k \in \mathbb{N}$.

5. a) Olkoon $A \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Joukko A on Lebesgue-mitallinen, jos

$$\mathcal{L}^n(E) = \mathcal{L}^n(E \cap A) + \mathcal{L}^n(E \setminus A)$$

kaikilla $E \subset \mathbb{R}^n$. Tässä $\mathcal{L}^n$ on Lebesguen (ulko)mitta.

b) Olkoon $m \in \mathbb{N}$, $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$, $A_1, A_2, \ldots, A_m \subset \mathbb{R}^n$ mitallisia joukkoja. Tällöin $\sum_{i=1}^{m} \alpha_i \chi_{A_i}(x)$ on yksinkertainen funktio, missä $\chi_{A_i}(x) = \begin{cases} 1, & \text{kun } x \in A_i \\ 0, & \text{kun } x \in \mathbb{R}^n \setminus A_i \end{cases}$.

Jos $f : \mathbb{R}^n \to \mathbb{R}$ on esitettävissä äärellisen monen mitallisen joukon karakteristisiden funktioiden (indikaattorifunktio) lineaarikombinaationa, f on yksinkertainen.

c) Jos f on yksinkertainen, niin sen integraali on

$$\int f \, d\lambda = \sum_{k=1}^{m} \alpha_k \mathcal{L}^n(A_k).$$

(Stokastiikassa: odotusarvo)

d) Olkoon $f \geq 0$ mitallinen. Tällöin, jos on jono yksinkertaisia funktioita $\{f_n\}_{n=1}^{\infty}$

s.e. $f(x) = \lim_{n\to\infty} f_n(x)$

ja $0 \leq f_1 \leq f_2 \leq \ldots \leq f$,

niin f:n integraali on

$$\int f \, d\mathcal{L}^n = \lim_{n\to\infty} \int f_n \, d\mathcal{L}^n .$$

e) Jos f on mitallinen ja $\int |f| \, d\mathcal{L}^n < \infty$,

niin f on integroituva ja

$$\int f \, d\mathcal{L}^n = \int f_+ \, d\mathcal{L}^n - \int f_- \, d\mathcal{L}^n ,$$

missä $f_+(x) = \max\{f(x), 0\}$ ja $f_-(x) = \max\{-f(x), 0\}$.

6. Olkoon $\Omega \subset \mathbb{R}^n$ mitallinen ja

$$L^{(p)}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \text{ mit. } \|f\|_p = \left(\int |f(x)|^p d\lambda\right)^{1/p} < \infty \right\}.$$

Osoitetaan, että kun $\frac{1}{p} + \frac{1}{q} = 1$ ja $f \in L^{(p)}$, $g \in L^{(q)}$, niin

$$\int |(fg)(x)| \, d\lambda \leq \left(\int |f(x)|^p d\lambda\right)^{1/p} \left(\int |g(x)|^q d\lambda\right)^{1/q} .$$

Todistus Olkoon $A = \left(\int_\Omega |f(x)|^p\right)^{1/p}$ ja

$B = \left(\int_\Omega |g(x)|^q \, d\lambda\right)$. Oletetaan, että

$0 < A, B < \infty$. Olkoon $\tilde{f}(x) = \frac{f(x)}{A}$

ja $\tilde{g}(x) = \frac{g(x)}{B}$. Tällöin

$$\int_\Omega |\tilde{f}(x)|^p \, d\lambda = 1 \quad \text{ja} \quad \int_\Omega |\tilde{g}(x)|^q \, d\lambda = 1 .$$

Käyttäen Youngin epäyhtälöä (kirjan

s. 13, (6)), saadaan

$$\tilde{f}(x)\tilde{g}(x) \leq \frac{\tilde{f}(x)^p}{p} + \frac{\tilde{g}(x)^q}{q} .$$

Integroimalla tämä saadaan

$$\int_\Omega \tilde{f}(x)\tilde{g}(x) \, d\lambda \leq \frac{1}{p} + \frac{1}{q} = 1$$

Lopulta

$$\int_\Omega (fg)(x) \, d\lambda = \|f\|_{L^{(p)}(\Omega)} \|g\|_{L^{(q)}(\Omega)} \int_\Omega \tilde{f}(x)\tilde{g}(x) \, d\lambda$$

$$\leq \|f\|_{L^p(\Omega)} \|g\|_{L^{(q)}(\Omega)} .$$

Tämä oli todistettava. $\square$