Mat-5.3741 Theory of Elasticity (5 cp) L **Spring 2007**
Stenberg/Juntunen

Exercise 4

Problem 1
Let $y = w(x)$ be a curve in xy-plane. Radius of curvature $\rho(x)$ is derived as the radius of the circle that fits best to $w(x)$ at point x. The conditions for this are that the circle and the curve have the same value $w(x)$, the same derivative $w'(x)$, and the same second derivative $w''(x)$ at the point x. Show that this gives

$$\rho(x) = \pm \frac{\left(1 + (w'(x))^2\right)^{3/2}}{w''(x)}.$$

Problem 2
Consider the rod problem : Find u such that

$$\begin{aligned} \frac{d}{dx}\left(EA\frac{du}{dx}\right) + f &= 0 \qquad\qquad 0 < x < L \\ u(0) &= 0 \\ EAu'(L) + ku(L) &= 0. \end{aligned}$$

Write the problem as a minimization and a variational problem. What boundary condition do you get in the limit $k \to \infty$ (by physical arguments)? Prove this a little bit more rigorously, let $|||\cdot|||_k$ (depends on $k > 0$) be the energy norm of the problem and let u_∞ be the candidate for the limit solution. Prove that it holds

$$|||u_\infty - u|||_k \leq k^{-1/2}|EAu'_\infty(L)|.$$

Problem 3
Consider the dynamics of a rod and show that the equation for the displacement $u(x,t)$ is

$$\frac{1}{c^2}\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2},$$

with $c =?$. Solve with Fourier series and plot (animate) the solution for the following boundary conditions
a) $u(0,t) = 0$, $u(L,t) = 0$
b) $u(0,t) = 0$, $EA\frac{du}{dx}(L,t) = 0$
and the initial conditions $u(x,0) = f(x)$, $\frac{du}{dt}(x,0) = g(x)$. Functions f and g you can choose as you wish.

Problem 4 **(home exercise)**
As derived in the book (p. 75) the normal stress in the beam varies linearly with y;

$$\sigma = -Eu''y.$$

In addition, we showed that

$$M = EIu'',$$

which gives

$$\sigma = \frac{M}{I}y.$$

Consider now a beam with a square cross section and rotated 45° w.r.t. the x- and y-axis (see figure). Show that for a fixed moment the maximal normal stress decreases if you take away material from the corners. Otimize the beam by computing the β for which the maximal stress is smallest.